Lifting the degeneracy between holographic CFTs

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Based on [2202.05261]
See also [2103.15830] with L. F. Alday, P. Ferrero, X. Zhou

Narrative

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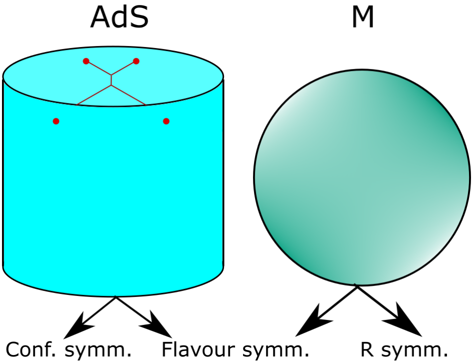

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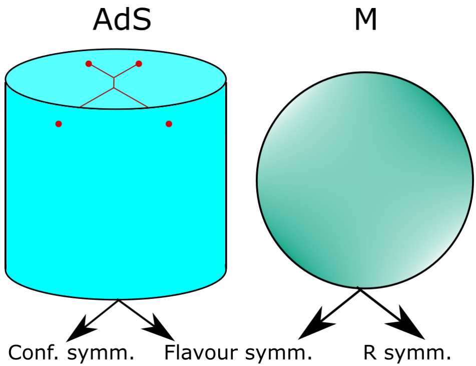

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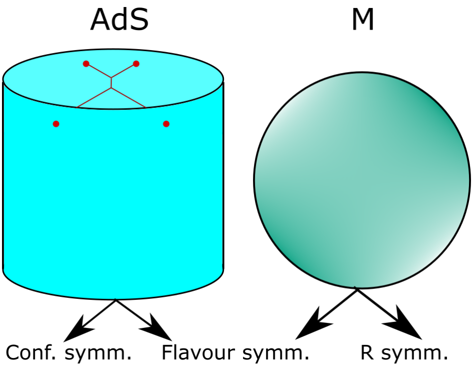

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More SUSY $\Rightarrow$ more protected operators.

The first holographic 4pt function

In $\mathcal{N} = 4$ SYM, single-trace superconformal primaries are p index traceless symmetric tensors of $SO(6)_R$ with $\Delta = p$ and $\ell = 0$:

$$\mathcal{O}_p(x, t) \equiv \mathcal{O}_{i_1 \dots i_p}(x) t^{i_1} \dots t^{i_p}, \quad t \cdot t = 0.$$

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In terms of $U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ and $\sigma = \frac{t_{13} t_{24}}{t_{12} t_{34}}, \tau = \frac{t_{14} t_{23}}{t_{12} t_{34}}$:

$$\langle \mathcal{O}_p \mathcal{O}_p \mathcal{O}_q \mathcal{O}_q \rangle = \left(\frac{t_{12}}{x_{12}^2} \right)^p \left(\frac{t_{34}}{x_{34}^2} \right)^q G(U, V; \sigma, \tau).$$

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Simplest case is $p = q = 2$ [D'Hoker, Freedman, Mathur, Matusis, Rastelli; 9903196] .

$$G(U, V; \sigma, \tau) = \int_{-i\infty}^{i\infty} \frac{ds dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t}{2} - 2} \mathcal{M}(s, t; \sigma, \tau) \Gamma \left[\tfrac{4-s}{2} \right]^2 \Gamma \left[\tfrac{4-t}{2} \right]^2 \Gamma \left[\tfrac{4-u}{2} \right]^2$$

$$\mathcal{M}(s, t; \sigma, \tau) = \mathcal{M}_s(s, t; \sigma, \tau) + \tau^2 \mathcal{M}_s \left(t, s; \tfrac{\sigma}{\tau}, \tfrac{1}{\tau} \right) + \sigma^2 \mathcal{M}_s \left(u, t; \tfrac{1}{\sigma}, \tfrac{\tau}{\sigma} \right)$$

$$\mathcal{M}_s(s, t; \sigma, \tau) = -\frac{60}{c_T} \frac{(t-4)(u-4) + (t-4)(s+2)\sigma + (u-4)(s+2)\tau}{s-2}$$

Degeneracy between theories

Background is $AdS_5 \times S^5$ for $SU(N)$.
Other gauge groups require $AdS_5 \times S^5/\mathbb{Z}_2$ [Witten; 9805112] .

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Not consistent with $SO(6)_R$ but $t = \left[\frac{y+\bar{y}}{2}, \frac{y-\bar{y}}{2i}\right]$ breaks it to $SU(3)_R$.
Single 4pt function turns into many:
$\mathcal{O}_2^4 \to \mathcal{O}_{[1,1]}^4 + \mathcal{O}_{[1,1]}^2 \mathcal{O}_{[2,0]} \mathcal{O}_{[0,2]} + \mathcal{O}_{[2,0]}^2 \mathcal{O}_{[0,2]}^2$.

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These belong to 4d $\mathcal{N} = 3$ SCFTs when $k = 3, 4, 6$.
Construction in [Garcia-Etxebarria, Regalado; 1512.06434] uses S-folds.

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Bulk now includes gauge fields switched on by $1/c_J$ leading to e.g.

$$\mathcal{M}_s^{I_1 I_2 I_3 I_4}(s, t; \alpha) = f^{I_1 I_2 J} f^{J I_3 I_4} \frac{6}{c_J} \frac{4 - u + \alpha(t + u - 8)}{s - 2}$$

along with $(\alpha - 1)^2 \mathcal{M}_s^{I_3 I_2 I_1 I_4} \left(t, s; \frac{\alpha}{\alpha - 1}\right)$, $\alpha^2 \mathcal{M}_s^{I_4 I_2 I_3 I_1} \left(u, t; \frac{1}{\alpha}\right)$.

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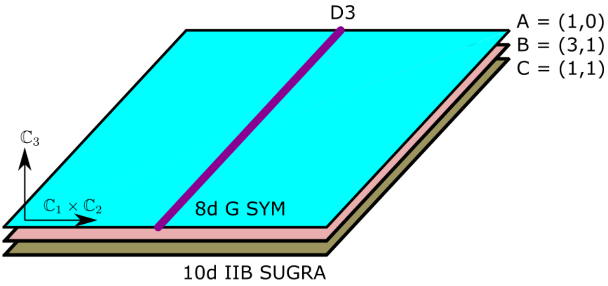
Tree-level correlators (any k) computed in [Alday, CB, Ferrero, Zhou; 2103.15830] .
First one-loop $k = 1$ correlator in [Alday, Bissi, Zhou; 2110.09861] .

Plan

1. Review of S-fold theories
2. Consequences for kinematics of local operators
3. Analytic bootstrap techniques
4. Anomalous dimensions at one loop
5. Future directions

4d $\mathcal{N} = 2$ backgrounds



G_F	ν
A_0	$1/3$
A_1	$1/2$
A_2	$2/3$
D_4	1
E_6	$4/3$
E_7	$3/2$
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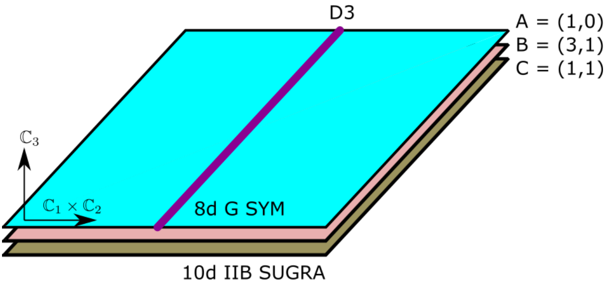

$$ds^2 = ds^2_{AdS_5} + d\phi^2 + \left(\frac{2-\nu}{2}\right)^2 \cos^2\phi \, d\theta^2 + \sin^2\phi \, ds^2_{S^3}$$

$$SO(6)_R \to SU(2)_L \times SU(2)_R \times U(1)_R$$

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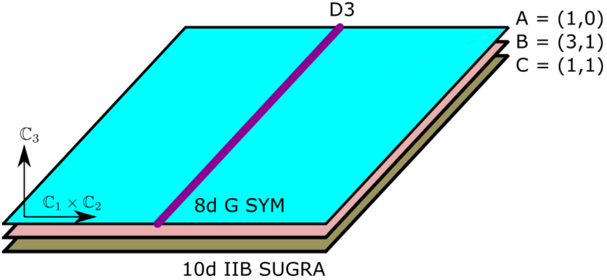

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Consider single-trace ops localized on 7-brane.

$$A^I_a(x,y) = \sum_{\mathfrak{M}} A^I_{\mathfrak{M}}(x) Y^{\mathfrak{M}}_a(y) \quad \Rightarrow \quad c_a^{b_1 \dots b_{p-1}} x_{b_1} \dots x_{b_{p-1}}$$

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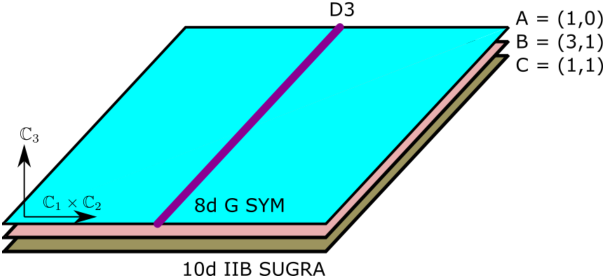

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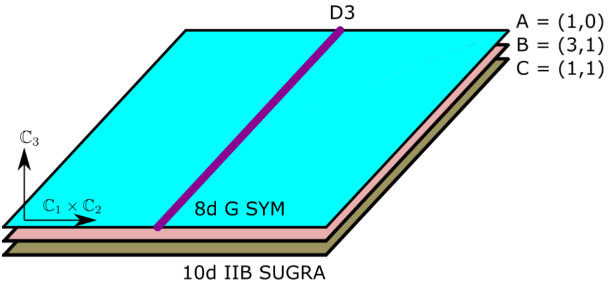
Primary and descendant spins are $(j_L, j_R) = \left(\frac{p-2}{2}, \frac{p}{2}\right) \oplus \left(\frac{p}{2}, \frac{p-2}{2}\right)$.

4d $\mathcal{N} = 2$ backgrounds with S-folds



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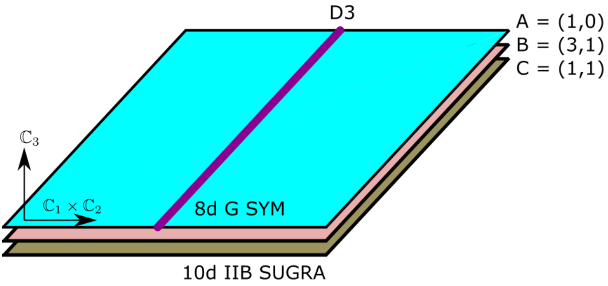

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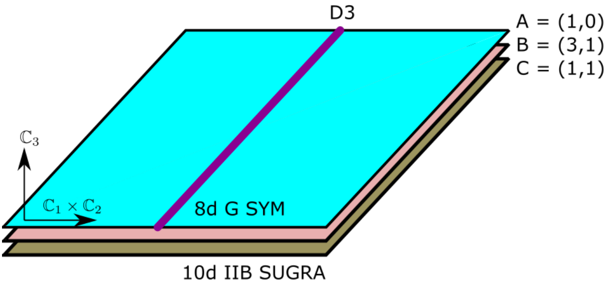

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Observe transformation of spherical harmonics

$$c^{\alpha_1\dots\alpha_{p-2};\bar{\alpha}_1\dots\bar{\alpha}_p} \mathrm{x}_{\alpha_1\bar{\alpha}_1}\dots\mathrm{x}_{\alpha_{p-2}\bar{\alpha}_{p-2}}, \quad \mathrm{x}_{\alpha\bar{\alpha}} = x_\mu \sigma^\mu_{\alpha\bar{\alpha}}$$

under $(\omega, \tilde{\omega}) \sim \left(\omega + \frac{2\pi}{k}, \tilde{\omega} - \frac{2\pi}{k}\right)$.

S-fold theory data

Result

If spherical harmonics for irrep p are labelled by $|m_L| \leq \frac{p-2}{2}$ and $|m_R| \leq \frac{p}{2}$, they survive the S-fold if and only if $k|2m_L$.

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Central charges known from [Giaocomelli, Meneghelli, Peelaers; 2007.00647] .

$\mathcal{S}^{(N)}_{G,k}$	G_F	c_J	c_T
$\mathcal{S}^{(N)}_{A_2,2}$	$USp(2) \times U(1)$	$\frac{3}{2}(3N+1)$	$90N^2 + \ldots$
$\mathcal{S}^{(N)}_{D_4,2}$	$USp(4) \times SU(2)$	$\frac{3}{2}(12N+1)$	$120N^2 + \ldots$
$\mathcal{S}^{(N)}_{E_6,2}$	$USp(8)$	$\frac{3}{2}(6N+1)$	$180N^2 + \ldots$
$\mathcal{S}^{(N)}_{A_1,3}$	$U(1)$	0	$120N^2 + \ldots$
$\mathcal{S}^{(N)}_{D_4,3}$	$SU(3)$	$3(6N+1)$	$180N^2 + \ldots$
$\mathcal{S}^{(N)}_{A_2,4}$	$SU(2)$	$3(6N+1)$	$180N^2 + \ldots$

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Related theories with different Coulomb branch : $\mathcal{S}^{(N)}_{G,k} \to \mathcal{T}^{(N)}_{G,k} \to \mathcal{S}^{(N-1)}_{G,k} \to \dots$

Correlation functions

Saturate all indices except the adjoint one for G_F.

$$\mathcal{O}^I_p(x; v, \bar{v}) \equiv \mathcal{O}^I_{\alpha_1 \dots \alpha_{p-2};\bar{\alpha}_1 \dots \bar{\alpha}_p}(x) v^{\alpha_1} \dots v^{\alpha_{p-2}} \bar{v}^{\bar{\alpha}_1} \dots \bar{v}^{\bar{\alpha}_p}$$

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Superblocks contain ≤ 20 bosonic blocks but they also solve $(z\partial_z - \alpha\partial_\alpha) \left. G \right|_{\alpha = z^{-1}} = 0$ [Dolan, Gallot, Sokatchev; 0405180] [Nirschl, Osborn; 0407060] .

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In terms of $\alpha = \frac{\bar{v}_{13}\bar{v}_{24}}{\bar{v}_{12}\bar{v}_{34}}, \beta = \frac{v_{13}v_{24}}{v_{12}v_{34}}$ and $U = z\bar{z}, V = (1-z)(1-\bar{z})$:

$$\left\langle \mathcal{O}^{I_1}_{j_1} \mathcal{O}^{I_2}_{j_1} \mathcal{O}^{I_3}_{j_2} \mathcal{O}^{I_4}_{j_2} \right\rangle = \left[\frac{\bar{v}_{12}}{x^2_{12}} \right]^{2j_1+2} \left[\frac{\bar{v}_{34}}{x^2_{34}} \right]^{2j_2+2} v^{2j_1}_{12} v^{2j_2}_{34} G^{I_1 I_2 I_3 I_4}(z, \bar{z}; \alpha, \beta).$$

Superblocks contain ≤ 20 bosonic blocks but they also solve $(z\partial_z - \alpha\partial_\alpha)\, G\big|_{\alpha=z^{-1}} = 0$ [Dolan, Gallot, Sokatchev; 0405180] [Nirschl, Osborn; 0407060] .

$$G(z, \bar{z}; \alpha) = \frac{z(1-\alpha\bar{z})f(\bar{z}) - \bar{z}(1-\alpha z)f(z)}{z - \bar{z}} + \frac{H(z, \bar{z}; \alpha)}{(1-\alpha z)^{-1}(1-\alpha\bar{z})^{-1}}$$

Long multiplets contribute one $U^{-1} g_{\Delta+2,\ell}(U, V)\mathcal{Y}_j(\alpha)$ to $H(U, V; \alpha)$.

⚠ Four such terms for $\mathcal{N} = 3$ [Lemos, Liendo, Meneghelli, Mitev; 1612.01536] .

Correlation functions

Blocks have simple expressions in 4d.

$$k_h(z) = z^h {}_2F_1(h, h; 2h; z), \qquad \mathcal{Y}_j(\alpha) = k_{-j}(\alpha^{-1})$$

$$g_{\Delta,\ell}(z, \bar{z}) = \frac{z\bar{z}}{z - \bar{z}} \left[k_{\frac{\Delta+\ell}{2}}(z) k_{\frac{\Delta-\ell-2}{2}}(\bar{z}) - (z \leftrightarrow \bar{z}) \right]$$

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Next step is to project out $U(1)_L$ components from

$$\langle \mathcal{O}_{j_1}(v_1) \mathcal{O}_{j_2}(v_2) \mathcal{O}_{j_3}(v_3) \rangle = C_{j_1,j_2,j_3} \, v_{12}^{j_1+j_2-j_3} v_{23}^{j_2+j_3-j_1} v_{31}^{j_3+j_1-j_2}.$$

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Use binomial theorem thrice on $v_{ij} = v_i^+ v_j^- - v_i^- v_j^+$ to get

$$\sum_{m_{ij}} \binom{j_1 + j_2 - j_3}{\frac{j_1+j_2-j_3}{2} + m_{12}} \binom{j_2 + j_3 - j_1}{\frac{j_2+j_3-j_1}{2} + m_{23}} \binom{j_3 + j_1 - j_2}{\frac{j_3+j_1-j_2}{2} + m_{31}} (-1)^{\#}$$
$$m_{12} - m_{31} = m_1, \quad m_{23} - m_{12} = m_2, \quad m_{31} - m_{23} = m_3.$$

Correlation functions

Result

To project $\langle \mathcal{O}_{j_1} \mathcal{O}_{j_2} \mathcal{O}_{j_3} \rangle$ *down to* $\langle \mathcal{O}_{j_1,m_1} \mathcal{O}_{j_2,m_2} \mathcal{O}_{j_3,m_3} \rangle$, *replace the tensor structure* $v_{12}^{j_1+j_2-j_3} v_{23}^{j_2+j_3-j_1} v_{31}^{j_3+j_1-j_2}$ *with*

$$\sqrt{\frac{(j_1+j_2-j_3)!(j_2+j_3-j_1)!(j_3+j_1-j_2)!}{(2j_1)!(2j_2)!(2j_3)!(j_1+j_2+j_3+1)!^{-1}}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

Correlation functions

Result

To project $\langle \mathcal{O}_{j_1} \mathcal{O}_{j_2} \mathcal{O}_{j_3} \rangle$ *down to* $\langle \mathcal{O}_{j_1,m_1} \mathcal{O}_{j_2,m_2} \mathcal{O}_{j_3,m_3} \rangle$*, replace the tensor structure* $v_{12}^{j_1+j_2-j_3} v_{23}^{j_2+j_3-j_1} v_{31}^{j_3+j_1-j_2}$ *with*

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Use this rule twice on each $C_{j_1,j_2,j_0} C_{j_3,j_4,j_0} \mathcal{Y}_{j_0}(\beta)$ appearing in tree-level 4pt functions of [Alday, CB, Ferrero, Zhou; 2103.15830] .

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Use this rule twice on each $C_{j_1,j_2,j_0} C_{j_3,j_4,j_0} \mathcal{Y}_{j_0}(\beta)$ appearing in tree-level 4pt functions of [Alday, CB, Ferrero, Zhou; 2103.15830] .

$$\mathcal{Y}_{j_0}(\beta) \propto (\partial_5 \cdot \partial_6)^{2j_0} \langle \mathcal{O}_{j_1}(v_1) \mathcal{O}_{j_2}(v_2) \mathcal{O}_{j_0}(v_5) \rangle \langle \mathcal{O}_{j_0}(v_6) \mathcal{O}_{j_3}(v_3) \mathcal{O}_{j_4}(v_4) \rangle$$

⚠ For groups broken to nonabelian subgroups, each harmonic polynomial will yield further polynomials instead of pure numbers.

Analytic bootstrap techniques

If $\Delta_{n,\ell} = \Delta^{(0)}_{n,\ell} + c_J^{-1}\gamma^{(1)}_{n,\ell} + \dots$ and $a_{n,\ell} = a^{(0)}_{n,\ell} + c_J^{-1} a^{(1)}_{n,\ell} + \dots,$

$$a_{n,\ell} g_{\Delta_{n,\ell},\ell} = a^{(0)}_{n,\ell} g_{\Delta^{(0)}_{n,\ell},\ell} + \frac{1}{c_J}\left[a^{(1)}_{n,\ell} + a^{(0)}_{n,\ell}\gamma^{(1)}_{n,\ell}\partial_\Delta\right] g_{\Delta^{(0)}_{n,\ell},\ell}$$
$$+ \frac{1}{c_J^2}\left[a^{(2)}_{n,\ell} + \left(a^{(1)}_{n,\ell}\gamma^{(1)}_{n,\ell} + a^{(0)}_{n,\ell}\gamma^{(2)}_{n,\ell}\right)\partial_\Delta + a^{(0)}_{n,\ell}\gamma^{(1)2}_{n,\ell}\frac{\partial^2_\Delta}{2}\right] g_{\Delta^{(0)}_{n,\ell},\ell}$$

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$$
\begin{aligned}
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&+ \frac{1}{c_J^2}\left[a^{(2)}_{n,\ell} + \left(a^{(1)}_{n,\ell}\gamma^{(1)}_{n,\ell} + a^{(0)}_{n,\ell}\gamma^{(2)}_{n,\ell}\right)\partial_\Delta + a^{(0)}_{n,\ell}\gamma^{(1)2}_{n,\ell}\frac{\partial^2_\Delta}{2}\right] g_{\Delta^{(0)}_{n,\ell},\ell}
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Leading log is special [Aharony, Alday, Bissi, Perlmutter; 1612.03891] .

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$$\frac{a_{n,\ell}}{\Delta - \Delta_{n,\ell}} = \frac{a^{(0)}_{n,\ell}}{\Delta - \Delta^{(0)}_{n,\ell}} + \frac{1}{c_J}\left[\frac{a^{(1)}_{n,\ell}}{\Delta - \Delta^{(0)}_{n,\ell}} + \frac{a^{(0)}_{n,\ell}\gamma^{(1)}_{n,\ell}}{(\Delta - \Delta^{(0)}_{n,\ell})^2}\right] + \frac{1}{c_J^2}\left[\frac{a^{(2)}_{n,\ell}}{\Delta - \Delta^{(0)}_{n,\ell}} + \frac{a^{(0)}_{n,\ell}\gamma^{(2)}_{n,\ell} + a^{(1)}_{n,\ell}\gamma^{(1)}_{n,\ell}}{(\Delta - \Delta^{(0)}_{n,\ell})^2} + \frac{a^{(0)}_{n,\ell}\gamma^{(1)2}_{n,\ell}}{(\Delta - \Delta^{(0)}_{n,\ell})^3}\right] + \dots$$

determined by Lorentzian inversion formula [Caron-Huot; 1702.00278] .

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Yields closed form holographic CFT data [Alday, Caron-Huot; 1711.02031] .

Analytic bootstrap techniques

Double log has a double discontinuity around $\bar{z} = 1$ defined by

$$dDisc[G(z, \bar{z})] = G(z, \bar{z}) - \frac{1}{2} G^{\circlearrowright}(z, \bar{z}) - \frac{1}{2} G^{\circlearrowleft}(z, \bar{z}).$$

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Extract $a^{(0)}_{n,\ell}$, $a^{(0)}_{n,\ell}\gamma^{(1)}_{n,\ell}$, etc by applying

$$c(\Delta,\ell) \propto \int_0^1 \frac{dz}{z^2}\frac{d\bar{z}}{\bar{z}^2}\left|\frac{z-\bar{z}}{z\bar{z}}\right|^2 g_{\ell+3,\Delta-3}(z,\bar{z}) dDisc[G(z,\bar{z})].$$

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Need > 1 correlators [Alday, Bissi; 1706.02388] [Aprile, Drummond, Heslop, Paul; 1706.02822] .

$$\left\langle a^{(0)}_{n,\ell}\gamma^{(1)2}_{n,\ell}\right\rangle \neq \left\langle a^{(0)}_{n,\ell}\gamma^{(1)}_{n,\ell}\right\rangle^2 / \left\langle a^{(0)}_{n,\ell}\right\rangle$$

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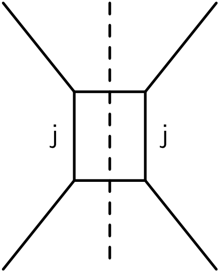

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[Alday, Chester, Raj; 2005.07175]

[Ferrero, Meneghelli; 2103.10440]

[Alday, Chester, Raj; 2107.10274]

[Alday, Bissi, Zhou; 2110.09861]

Zeroth order

For $\langle \mathcal{O}_j \mathcal{O}_j \mathcal{O}_j \mathcal{O}_j \rangle$,

$$G^{I_1 I_2 I_3 I_4}(U, V; \alpha, \beta) = \delta^{I_1 I_2} \delta^{I_3 I_4} + (\alpha U)^{2j+2} \beta^{2j} \delta^{I_1 I_3} \delta^{I_2 I_4} + \frac{[(\alpha - 1)U/V]^{2j+2}}{(\beta - 1)^{-2j}} \delta^{I_1 I_4} \delta^{I_2 I_3}$$

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$$H^{I_1 I_2 I_3 I_4}(z, \bar{z}; \alpha, \beta) = \beta^{2j} \sum_{l=0}^{2j} \alpha^l \sum_{m=0}^{2j-l} z^{l+m+1} \bar{z}^{2j-m+1} \delta^{I_1 I_3}\delta^{I_2 I_4}$$

$$+ \frac{(1-\beta)^{2j}}{(z-1)(\bar{z}-1)} \sum_{l=0}^{2j} (1-\alpha)^l \sum_{m=0}^{2j-l} \left(\frac{z}{z-1}\right)^{l+m+1} \left(\frac{\bar{z}}{\bar{z}-1}\right)^{2j-m+1} \delta^{I_1 I_4}\delta^{I_2 I_3}.$$

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Pull out $\mathcal{Y}_0(\alpha)\mathcal{Y}_0(\beta)$ and $\mathbf{R}_a$ component. Referring to [Cvitanovič; 08] ,
$P_a^{I_1 I_2 | I_3 I_4} \delta^{I_1 I_4} \delta^{I_2 I_3} = dim(G_F) P_a^{I_1 I_2 | I_3 I_4} P_{sing}^{I_1 I_4 | I_2 I_3} = dim(G_F)(F_t)_a^{sing}$.

Zeroth order

For $\langle \mathcal{O}_j \mathcal{O}_j \mathcal{O}_j \mathcal{O}_j \rangle$,

$$G^{l_1 l_2 l_3 l_4}(U, V; \alpha, \beta) = \delta^{l_1 l_2}\delta^{l_3 l_4} + (\alpha U)^{2j+2}\beta^{2j}\delta^{l_1 l_3}\delta^{l_2 l_4} + \frac{[(\alpha - 1)U/V]^{2j+2}}{(\beta - 1)^{-2j}}\delta^{l_1 l_4}\delta^{l_2 l_3}$$

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$$+ \frac{(1-\beta)^{2j}}{(z-1)(\bar{z}-1)} \sum_{l=0}^{2j} (1-\alpha)^l \sum_{m=0}^{2j-l} \left(\frac{z}{z-1}\right)^{l+m+1} \left(\frac{\bar{z}}{\bar{z}-1}\right)^{2j-m+1} \delta^{l_1 l_4}\delta^{l_2 l_3}.$$

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$$H_a(z, \bar{z}) = \frac{dim(G_F)(F_t)_a^{sing}}{2j+1} \frac{z\bar{z}}{z - \bar{z}} \left[\sum_{l=0}^{2j} \frac{z^{2j+1}\bar{z}^l - \bar{z}^{2j+1}z^l}{l+1} \right.$$

$$\left. - \sum_{l=0}^{2j} \frac{1}{l+1} \left(\frac{z^{2j+1}\bar{z}^l}{(z-1)^{2j+2}(\bar{z}-1)^{l+1}} - \frac{\bar{z}^{2j+1}z^l}{(\bar{z}-1)^{2j+2}(z-1)^{l+1}} \right) \right]$$

Zeroth order

$$c_a(h,\bar{h}) = \frac{r(\bar{h})^2}{4\pi^2}\int_0^1 \frac{dz}{z^2} k_{1-h}(z) \int_0^1 \frac{d\bar{z}}{\bar{z}^2} \frac{k_{\bar{h}}(\bar{z})}{\bar{h}-\frac{1}{2}} dDisc[(\bar{z}-z)H_a(z,\bar{z})]$$

$$h \equiv \frac{\Delta-\ell}{2}, \quad r(h) \equiv \frac{\Gamma(h)^2}{\Gamma(2h-1)}, \quad \bar{h} \equiv \frac{\Delta+\ell+2}{2}.$$

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Result

Defining $h = 2j + n + 2, \bar{h} = 2j + n + \ell + 3,$ *GFF coefficients are*

$$\left\langle a^{(0)} \right\rangle^{(j)}_{a,n,\ell} = \frac{2dim(G_F)(F_t)^{sing}_a}{(2j+1)!^4} \frac{(h-2j-1)_{4j+2} r(h)}{(\bar{h}-2j-1)^{-1}_{4j+2} r(\bar{h})^{-1}} \frac{\bar{h}(\bar{h}-1) - h(h-1)}{h(h-1)\bar{h}(\bar{h}-1)}$$

for $\langle \mathcal{O}_j \mathcal{O}_j \mathcal{O}_j \mathcal{O}_j \rangle$

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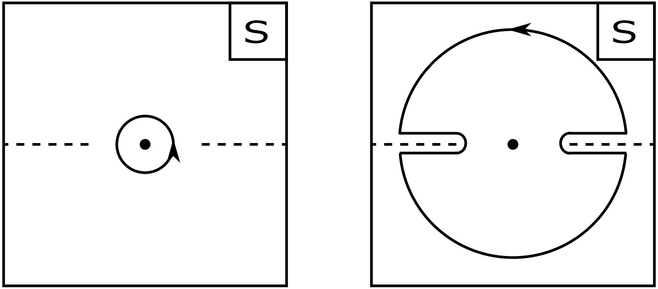
$$\left\langle a^{(0)} \right\rangle^{(j)}_{a,n,\ell} = \frac{2dim(G_F)(F_t)^{sing}_a}{(2j+1)!^4} \frac{(h-2j-1)_{4j+2} r(h)}{(\bar{h}-2j-1)^{-1}_{4j+2} r(\bar{h})^{-1}} \frac{\bar{h}(\bar{h}-1)-h(h-1)}{h(h-1)\bar{h}(\bar{h}-1)}$$

for $\langle \mathcal{O}_j \mathcal{O}_j \mathcal{O}_j \mathcal{O}_j \rangle$

$$\left\langle a^{(0)} \right\rangle^{(j,m)}_{a,n,\ell} = (1+\delta_{m,0}) \left(j + \frac{1}{2} \right) \left\langle a^{(0)} \right\rangle^{(j)}_{a,n,\ell}$$

for $\langle \mathcal{O}_{j,m} \mathcal{O}_{j,-m} \mathcal{O}_{j,m} \mathcal{O}_{j,-m} \rangle$.

Trustworthiness of low spins



Soft behaviour in Regge limit ($s \to \infty$ for fixed t) is required to drop the arcs.

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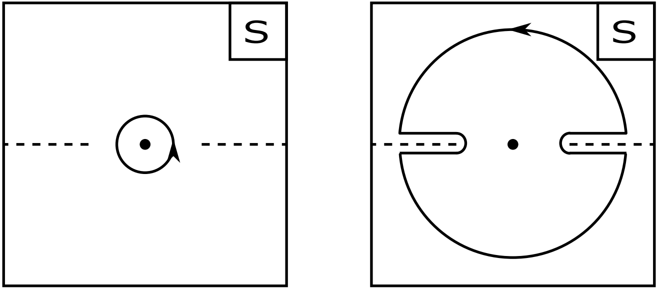

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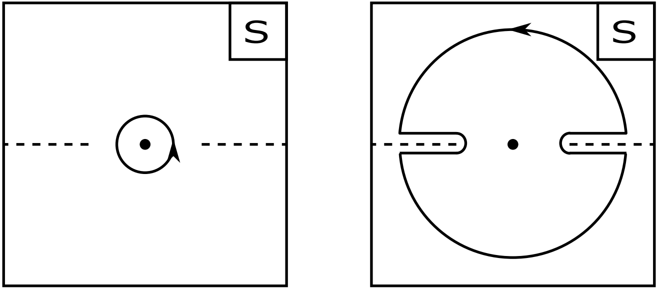

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Critical spin from $G(w\sigma, w/\sigma) \sim w^{1-\ell_*}$ [Maldacena, Shenker, Stanford; 1503.01409] .

First order

Mellin amplitudes in [Alday, CB, Ferrero, Zhou; 2103.15830] expressed using "super Witten diagram" $\mathcal{S}_j(s,t;\alpha)$.

$$\mathcal{M}^{I_1 I_2 I_3 I_4}(s,t;\alpha,\beta) = f^{I_1 I_2 J} f^{J I_3 I_4} \sum_j C_{j_1,j_2,j} C_{j_3,j_4,j} \mathcal{Y}_j(\beta) \mathcal{S}_j(s,t;\alpha) + (t-channel) + (u-channel) + (contact)$$

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For $\langle \mathcal{O}_0 \mathcal{O}_0 \mathcal{O}_j \mathcal{O}_j \rangle$, this is $\left[1 - \alpha(1 + \widehat{U} - \widehat{V}) + \alpha^2 \widehat{U} \right] \circ \widetilde{\mathcal{M}}^{I_1 I_2 I_3 I_4}(s,t)$ with

$$\widetilde{\mathcal{M}}^{I_1 I_2 I_3 I_4}(s,t) = -\frac{24}{c_J (2j)!} \left[\frac{f^{I_1 I_2 J} f^{J I_3 I_4}}{(s-2)(u-2j-4)} - \frac{f^{I_1 I_4 J} f^{J I_2 I_3}}{(t-2j-2)(u-2j-4)} \right].$$

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Only the pole at $t = 2j+2$ can give a double discontinuity.

$$H_a(U,V) = \int_{-i\infty}^{i\infty}\frac{dsdt}{(4\pi i)^2}U^{\frac{s}{2}}V^{\frac{t}{2}-j-2}\widetilde{\mathcal{M}}_a(s,t)\Gamma\left[\tfrac{4-s}{2}\right]\Gamma\left[\tfrac{4j+4-s}{2}\right]\Gamma\left[\tfrac{2j+4-t}{2}\right]^2\Gamma\left[\tfrac{2j+6-u}{2}\right]^2$$

First order

Result

Defining $h = n + 2, \bar{h} = n + \ell + 3$ *and*

$$\frac{R_b(h)}{r(h)} = \frac{\Gamma(h - b - 1)}{\Gamma(h + b + 1)},$$

weighted averages of anomalous dimensions are

$$\left\langle a^{(0)}\gamma^{(1)} \right\rangle^{(j,m)}_{a,n,\ell} = \left\langle a^{(0)}\gamma^{(1)} \right\rangle^{(j)}_{a,n,\ell}$$

$$= (-1)^{2j} \frac{12 G_F^\vee (F_t)_a^{adj}}{(2j)!(2j+1)!} R_{-2j-2}(h) R_{-1}(\bar{h}).$$

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These are sums of up to $n + 1$ true anomalous dimensions due to

$$[\mathcal{O}_0\mathcal{O}_0]_n, [\mathcal{O}_{1/2}\mathcal{O}_{1/2}]_{n-1}, \dots, [\mathcal{O}_{n/2}\mathcal{O}_{n/2}]_0.$$

Application to S-folds

Need upper left entry of M^2 where $M = Q\, diag(\gamma_1, \dots, \gamma_{n+1}) Q^T$.

$$\left\langle a^{(0)} \gamma^{(1)2} \right\rangle^{(k=1)}_{a,n,\ell} = \sum_{2j=0}^{n} \frac{\left\langle a^{(0)} \gamma^{(1)} \right\rangle^{(j)2}_{a,n,\ell}}{\left\langle a^{(0)} \right\rangle^{(j)}_{a,n-2j,\ell}}$$

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$$\left\langle a^{(0)} \gamma^{(1)2} \right\rangle^{(k=4)}_{a,n,\ell} = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{2\lfloor j/2 \rfloor + 1}{2j+1} \frac{\left\langle a^{(0)} \gamma^{(1)} \right\rangle^{(j)2}_{a,n,\ell}}{\langle a^{(0)} \rangle^{(j)}_{a,n-2j,\ell}}.$$

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Double discontinuity now follows as

$$\mathcal{G}_a(z,\bar{z}) = \sum_{n=0}^{\infty} \sum_{\ell} \frac{1}{8} \left\langle a^{(0)}\gamma^{(1)2}\right\rangle_{a,n,\ell} \frac{(z-\bar{z})z^2\bar{z}^2}{(1-z)^3(1-\bar{z})^3} g_{6+2n+\ell,\ell}(1-z,1-\bar{z}).$$

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Basis over $\mathbb{C}(z, \bar{z})$ of $1, \log(z), Li_2(1-z), Li_2(1-z^{-1}), (z \leftrightarrow \bar{z})$ is often enough [Aprile, Drummond, Heslop, Paul; 1706.02822] [Alday, Caron-Huot; 1711.02031] .

Dealing with the infinite sum

$$\mathcal{G}_a(x,y) = \frac{[6 G_F^\vee (F_t)_a^{adj}]^2}{dim(G_F)(F_t)_a^{sing}} \sum_{n=0}^{\infty} \mathcal{H}_n(x,y), \qquad \mathcal{H}_n(x,y) \sim y^n$$

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Easy part is $k_h(z)k_{\bar{h}}(\bar{z})$, hard part is $k_h(\bar{z})k_{\bar{h}}(z)$.

$$k_h\left(\frac{1}{x+1}\right) R_{-2j-2}(h) \sum_\ell R_{2j+1}(\bar{h})\bar{h}(\bar{h}-1)(-1)^{n+\ell} k_{\bar{h}}(-y)$$

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Resummation understood in [Simmons-Duffin; 1612.08471] .

$$\sum_{\ell=0}^\infty R_b(-b+\ell) k_{-b+\ell}\left(\tfrac{1}{x+1}\right) = \Gamma(-b)^2 x^b$$

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$$\sum_{\ell=0}^\infty R_b(h_0+\ell)k_{h_0+\ell}\left(\tfrac{1}{x+1}\right) = \Gamma(-b)^2\left[x^b + \sum_{m=0}^\infty \partial_m\left(x^m\mathcal{A}_{b,-m-1}(h_0)\right)\right]$$

$$\mathcal{A}_{l,m}(h_0) = -\frac{(l+h_0)(m+h_0)}{l+m+1}\frac{R_l(h_0)R_m(h_0)}{\Gamma(-l)^2r(h_0)^2\Gamma(-m)^2}$$

Main results

For even spin, $\log x$ part of $\mathcal{G}_a(x, y)$ looks like

$$x^2(-1 + 10y + 18y^2) + \frac{x^3}{3}(5 + 148y + 1017y^2 + 1080y^3) + O(x^4) \quad (k = 1)$$

$$\frac{x^2}{105}(-105 - 420y + 1827y^2 + 1784y^3 + \dots)$$
$$-\frac{x^3}{105}(-105 + 840y - 42777y^2 + 7744y^3 + \dots) + O(x^4) \qquad (k = 2).$$

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$$-\frac{x^3}{105}(-105+840y-42777y^2+7744y^3+\dots)+O(x^4)\qquad (k=2).$$

Term with x^2 can give non-averaged anomalous dimension.

$$\gamma^{(2)}_{a,0,\ell}=144\frac{[G_F^\vee(F_t)_a^{adj}/dim(G_F)(F_t)_a^{sing}]^2}{\ell(\ell+1)^2(\ell+4)^2(\ell+5)}\frac{\ell^4+6\ell^3-25\ell^2-150\ell-96}{(\ell+1)(\ell+4)}\quad (k=1)$$

$$\gamma^{(2)}_{a,0,\ell}=\frac{144}{5}\frac{[G_F^\vee(F_t)_a^{adj}/dim(G_F)(F_t)_a^{sing}]^2}{(\ell)_6(\ell+1)(\ell+4)}\left[\frac{5\ell^6+55\ell^5+195\ell^4+205\ell^3-896\ell^2-3980\ell-2784}{(\ell+1)(\ell+4)}+\dots\right]\quad (k=2)$$

Outlook

- Loop anomalous dimensions can also help distinguish which CFT saturates a numerical bootstrap bound.
- Which other S-fold theories are within reach?
- Resummed lightcone bootstrap and inversion formula appear to be more powerful when used together.
- In $\mathcal{N} = 4$ SYM, fixed small spin can be brought under control too [Alday, Chester, Hansen; 2110.13106] .
- One can also explore more recent variations of the AdS unitarity method [Meltzer, Perlmutter, Sivaramakrishnan; 1912.09521] .