Coupled Minimal Models (Irrationally) Revisited

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[2211.16503] with A. Antunes

When does the ε expansion work best?

$$\mathit{CFT}_{UV} \xrightarrow{\text{Relevant } \mathcal{O}} \mathit{CFT}_{IR}$$

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Free scalar $\quad CFT_{UV}$

$$CFT_{UV} \xrightarrow{\lambda\phi^4} CFT_{IR}$$

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$N \times$ Free scalar $\quad CFT_{UV}$

$$\downarrow \; \lambda_{ijkl}\phi^i\phi^j\phi^k\phi^l$$

CFT_{IR}

Zoo of new fixed points in

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$$S_1 = \sum_{i=1}^{N} S^i_{Ising} + g \int d^d x \sum_{i<j} \epsilon_i \epsilon_j$$

$$S_2 = \sum_{i=1}^{N} S^i_{q-Potts} + g \int d^2 x \sum_{i<j} \epsilon_i \epsilon_j$$

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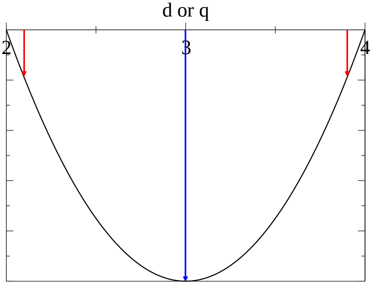

Questions about these models

- S_1: Is fixed point in $d = 3$ more stable than $O(3)$? [Aharony; 73]
- Yes. [Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi; 20]
- S_2: Is fixed point for $q = 3$ rational? [Dotsenko, Jacobsen, Lewis, Picco; 98]
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Extended chiral algebra	Exact methods	No known methods

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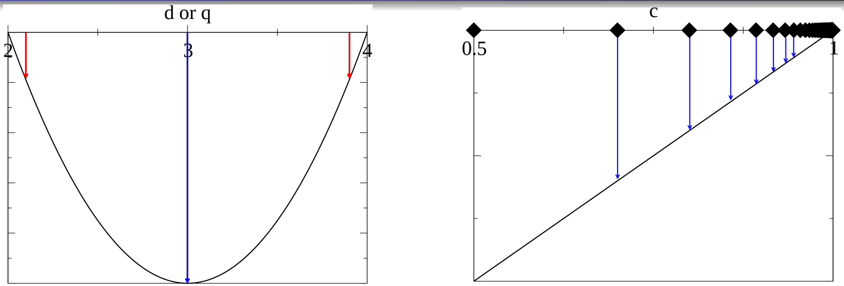
No literature on irrational, unitary CFTs with discrete spectrum and only Virasoro symmetry (irrational sigma models have higher symmetry like $\mathcal{N} = 2$)!

Minimal models at large m



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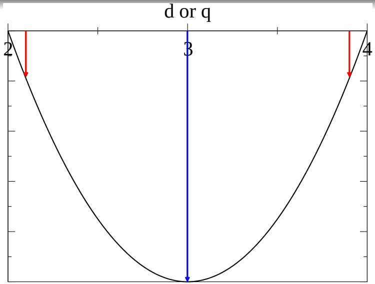

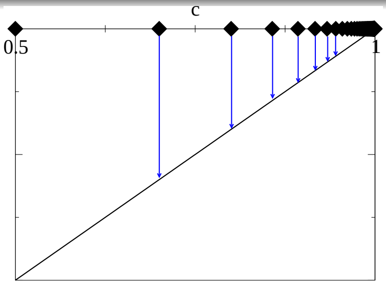

From $c = 1 - \frac{6}{m(m+1)}$ and $h_{(r,s)} = \frac{[(m+1)r-ms]^2-1}{4m(m+1)}$,
$\phi^i_{(1,2)}$ has weight $\frac{1}{4} - O(m^{-1})$, $\phi^i_{(1,3)}$ has weight $1 - O(m^{-1})$.

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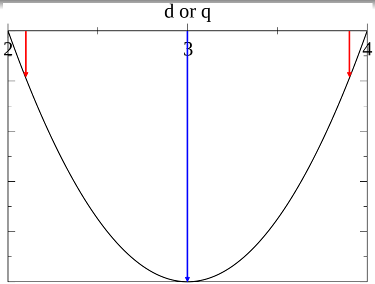

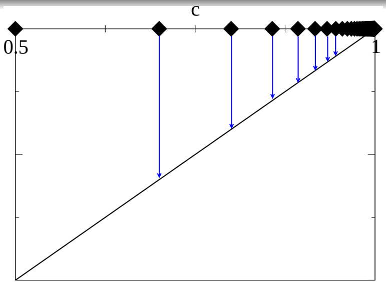

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$$S = \sum_{i=1}^{N} S^i_m + g_\epsilon \int d^2x \, N^{-\frac{1}{2}} \sum_{i=1}^{N} \phi^i_{(1,3)}$$

$$+ g_\sigma \int d^2x \binom{N}{4}^{-\frac{1}{2}} \sum_{i<j<k<l} \phi^i_{(1,2)} \phi^j_{(1,2)} \phi^k_{(1,2)} \phi^l_{(1,2)}$$

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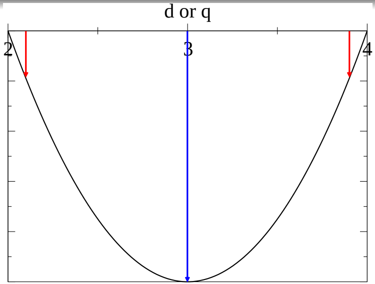

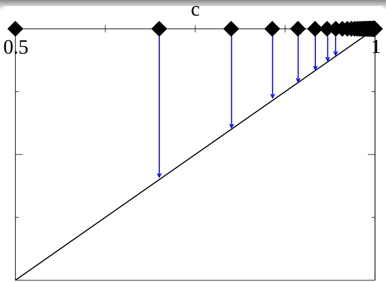

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Renormalization group

Use one loop conformal perturbation theory [Zamolodchikov; 87] .

$$\beta_\sigma = \frac{6}{m} g_\sigma - 4\pi \sqrt{\frac{3}{N}} g_\sigma g_\epsilon - 6\pi \binom{N-4}{2} \binom{N}{4}^{-\frac{1}{2}} g_\sigma^2$$

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For $P = 3N^4 - 53N^3 + 357N^2 - 1069N + 1194$, $Q = 3(N-4)(N-5)$,

$$(g_\sigma^*, g_\epsilon^*) = (0,0), \quad (g_\sigma^*, g_\epsilon^*) = \left(0, \frac{2\sqrt{3}}{m\pi}\right)$$
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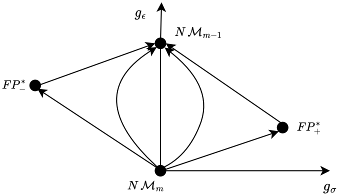

Dimensions of σ, ϵ for $N = 4$ become $\Delta = 2 \pm \frac{2\sqrt{6}}{m}$.

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$$bg \langle V(z_1) V(z_2) \rangle = \left\langle \bar{\partial} T(z_1) V(z_2) \right\rangle = g \int d^2 z \left\langle \bar{\partial} T(z_1) V(z_2) \sigma(z) \right\rangle$$

Agrees with two loop calculation [CB, Rastelli, Rychkov, Zan; 17] .

Lifting of currents

T^i goes with $V^i \equiv \sum_{(j<k<l)\neq i}[\partial\phi^i]\phi^j\phi^k\phi^l - \frac{1}{4}\partial[\phi^i\phi^j\phi^k\phi^l]$ yielding

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$$T_4 \left|0\right\rangle = \left[\sum_i L^i_{-4} - \frac{5}{3}\sum_i (L^i_{-2})^2 + \frac{18}{N-1}\sum_{i<j} L^i_{-2}L^j_{-2} \right] \left|0\right\rangle \Rightarrow$$

$$\gamma[T_4] = (g_\sigma^* \pi)^2 \frac{5N+22}{2N(N-1)}$$

because rows of 1×2 matrix $\left\langle T_4^I V_3^J \sigma \right\rangle$ are linearly independent.

Check over 1 CPU day

The number of (currents, potential divergences):

ℓ \ N	4	5	6	7
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- Should determine conformal window non-perturbatively.
- For $N = 4$, check if W-algebra is $\mathcal{W}(2, 6)$.
- Consider S_N breaking flows e.g. $\mathbb{Z}_N$ as in 3d [LeClair, Ludwig, Mussardo; 97] .
- Couple $\mathcal{W}[\mathfrak{d}_n]$ minimal models [Dotsenko, Nguyen, Santachiara; 01] .