Making $AdS_3 \times S^3$ more like $AdS_5 \times S^5$

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ICTP-SAIFR

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Based on [2408.17420] with R. S. Pitombo
Also in progress with M. Nocchi and R. S. Pitombo

Two classic examples of holography

First examples of AdS_5/CFT_4 and AdS_3/CFT_2 are both from [Maldacena; hep-th/9711200] .

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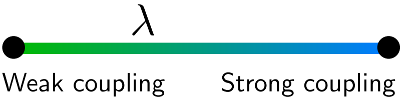

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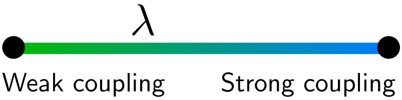

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Compute with SUSY localization [Chester, Pufu; 2003.08412]

$$M = \frac{1}{N^2}\left[\frac{1}{(s-2)(t-2)(u-2)} + \frac{b_1}{\lambda^{3/2}} + \dots\right] + \frac{1}{N^4}\left[\lambda^{1/2} b_2 + M_{loop} + \dots\right]$$

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$$M = \frac{1}{N^2}\frac{1}{stu} + \frac{1}{N^2}\sum_{a,b=0}^{\infty} \frac{\sigma_2^a \sigma_3^b}{\lambda^{a+\frac{3}{2}b+\frac{3}{2}}}\left(\alpha_{a,b}^{(0)} + \frac{\alpha_{a,b}^{(1)}}{\sqrt{\lambda}} + \dots\right) + O(N^{-4})$$

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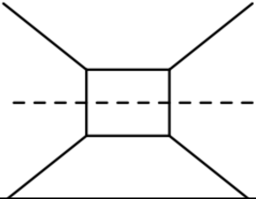

Some loops and stringy corrections are known with higher KK modes [Fardelli, Hansen, Silva; 2308.03683] .

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Put D1-D5 on $\mathbb{R}^6 \times M_4$ for $AdS_3 \times S^3$. Setup can have both R-R and NS-NS flux. The latter is much nicer! [Maldacena, Ooguri; hep-th/0001053]

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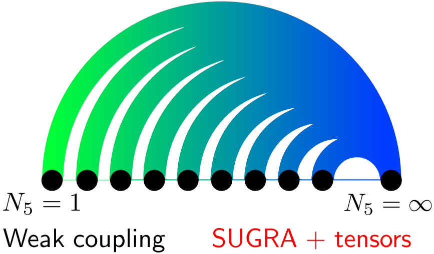

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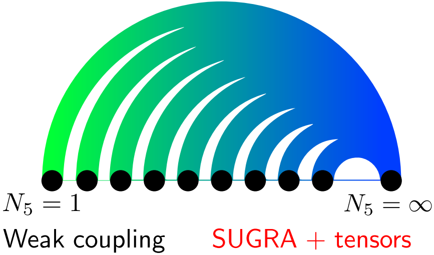

$S^3 \times S^1 \Rightarrow$ large $\mathcal{N} = 4$ Virasoro

$T^4 \Rightarrow$ contraction $\rightarrow PSU(1,1|2)^2$

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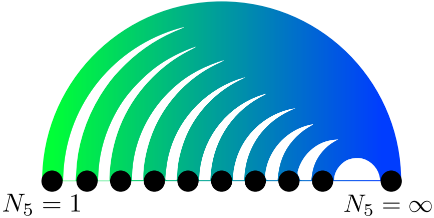

Weak coupling   SUGRA + tensors

→ Tensor with Liouville [Eberhardt, Gaberdiel; 1903.00421]

→ $Sym^N(M_4)/S_N$ for $N = N_1 N_5$ [Eberhardt, Gaberdiel, Gopakumar; 1812.01007]

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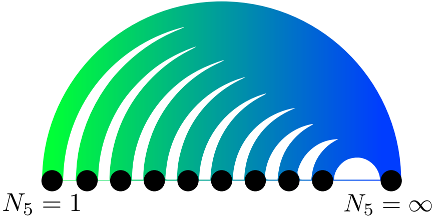

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First correction around flat space found in [Chester, Zhong; 2412.06429] .

The single-trace spectrum (AdS_5)

$\mathcal{N} = 4$ SYM operators below $\Delta \sim \lambda^{1/4}$ should be very simple! Just operators dual to single supergravity particles (e.g. $\text{Tr}(X^I X^J)$) and their composites (e.g. $\text{Tr}(X^I X^J)\text{Tr}(X^I X^J)$). What representation are they in?

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Slow way: Sort solutions of linearized EOMs into half-BPS $B\bar{B}[0,0]_k^{[0,k,0]}$ multiplets [Kim, Romans, van Nieuwenhuizen; 1985] .

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- $[0,k,0]$ → k symmetrized R indices
- subscript k → Scaling dimension k
- $[0,0]$ → Lorentz scalar

Fast way: Just use all multiplets where spin is at most 2! Very short list because $PSU(2,2|4)$ has 16 supercharges.

$$S_k(x,t) = t_{I_1} \dots t_{I_k} S_k^{I_1 \dots I_k}(x)$$

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$V_k^+(v,\bar{v})$ with $k+1$ v's and $k-1$ $\bar{v}$'s
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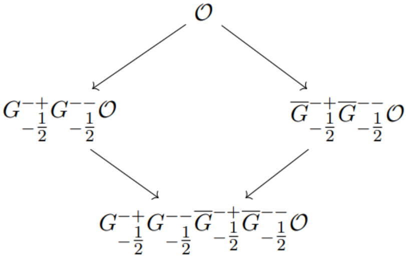

Types of four-point functions

[Rastelli, Zhou; 1608.06624]

$$\langle S_{k_1} S_{k_2} S_{k_3} S_{k_4} \rangle$$

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$\left\langle s^{I_1}_{k_1} s^{I_2}_{k_2} s^{I_3}_{k_3} s^{I_4}_{k_4} \right\rangle$ $\left\langle s^{I_1}_{k_1} s^{I_2}_{k_2} \sigma_{k_3} \sigma_{k_4} \right\rangle$ $\langle \sigma_{k_1} \sigma_{k_2} \sigma_{k_3} \sigma_{k_4} \rangle$ $\langle V^+_{k_1} V^+_{k_2} V^+_{k_3} V^+_{k_4} \rangle$

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[Rastelli, Roumpedakis, Zhou; 19] [Wen, Zhang; 21] [CB, Pitombo; 24]

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Form of a four-point function

Extract kinematic factor $\mathbf{K}$ depending on positions x_i and polarizations t_i or $(v_i, \bar{v}_i)$. Interested in cross ratio dependence,

$$\langle \mathcal{O}_{k_1} \mathcal{O}_{k_2} \mathcal{O}_{k_3} \mathcal{O}_{k_4} \rangle = \mathbf{K} G(z, \bar{z}, \alpha, \bar{\alpha}).$$

In all cases,

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}).$$

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For $AdS_5 \times S^5$,

$$\sigma = \frac{t_{13} t_{24}}{t_{12} t_{34}} = \alpha\bar{\alpha}, \quad \tau = \frac{t_{14} t_{23}}{t_{12} t_{34}} = (1-\alpha)(1-\bar{\alpha}).$$

For $AdS_3 \times S^3$,

$$\alpha = \frac{v_{13} v_{24}}{v_{12} v_{34}}, \quad \bar{\alpha} = \frac{\bar{v}_{13} \bar{v}_{24}}{\bar{v}_{12} \bar{v}_{34}}.$$

The bootstrap approach

Make an ansatz based on AdS/CFT and fix coefficients using superconformal Ward identity [Dolan, Gallot, Sokatchev; hep-th/0405180]

$$\left(z\partial_z - \alpha\partial_\alpha\right) G\Big|_{\alpha=z^{-1}} = 0, \quad \left(\bar{z}\partial_{\bar{z}} - \bar{\alpha}\partial_{\bar{\alpha}}\right) G\Big|_{\bar{\alpha}=\bar{z}^{-1}} = 0.$$

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Make an ansatz based on AdS/CFT and fix coefficients using superconformal Ward identity [Dolan, Gallot, Sokatchev; hep-th/0405180]

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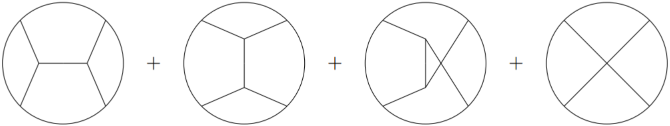

Setup in [Rastelli, Roumpedakis, Zhou; 1905.11983] uses exchange Witten diagrams and contact terms

$$G^{I_1 I_2 I_3 I_4}_{k_1,k_2,k_3,k_4}(z,\bar{z},\alpha,\bar{\alpha}) = \delta^{I_1 I_2}\delta^{I_3 I_4}\, G^{(s)}_{k_1,k_2,k_3,k_4}(z,\bar{z},\alpha,\bar{\alpha}) + \text{crossed}$$

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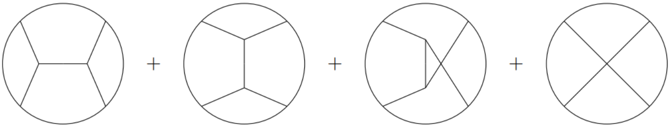

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$$\begin{aligned} s_1 \times s_1 &= V_1, \\ s_2 \times s_2 &= V_1 + V_3 + \sigma_2, \\ s_3 \times s_3 &= V_1 + V_3 + V_5 + \sigma_2 + \sigma_4, \ldots \end{aligned}$$

Inputs vs outputs

OPE coefficients can be fixed as outputs for $AdS_5 \times S^5$ but not for $AdS_3 \times S^3$. Use SUGRA results [Arutyunov, Pankiewicz, Theisen; hep-th/0007601] .

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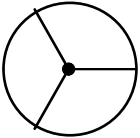

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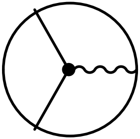

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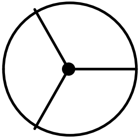

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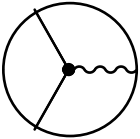

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Now fix coefficients in $\mathcal{C}(z, \bar{z}, \alpha, \bar{\alpha})$ by parameterizing this piece and $\mathcal{S}_{\mathcal{O}}(z, \bar{z}, \alpha, \bar{\alpha})$ in terms of $\bar{D}$ functions [Dolan, Osborn; hep-th/0011040] .

Enter Mellin space

For a correlator $G(U, V)$ (which may depend on $\alpha, \bar{\alpha}$):

$$G(U, V) = \int_{-i\infty}^{i\infty} \frac{ds dt}{(4\pi i)^2} U^{\frac{s}{2}+a_s} V^{\frac{t}{2}+a_t} \mathcal{M}(s, t)\Gamma(s, t)$$

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$$\underbrace{\mathcal{W}_{\Delta,\ell}(U, V)}_{\text{Polynomial}} = \underbrace{G_{\Delta,\ell}(U, V)}_{\text{Exponential}} + \sum_{n=0}^{\infty} \beta_n G_{2\Delta_\phi+2n+\ell,\ell}(U, V)$$

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Gives families of tree-level four-point functions in $\mathcal{N} = 4$ SYM and other holographic CFTs [Alday, Zhou; 2006.12505] [Alday, CB, Ferrero, Zhou; 2103.15830] .

$$U\partial_U G(U, V) \mapsto \left(\frac{s}{2} + a_s\right) \mathcal{M}(s, t), \quad V\partial_V G(U, V) \mapsto \left(\frac{t}{2} + a_t\right) \mathcal{M}(s, t),$$

$$U^m V^n G(U, V) \mapsto \frac{\Gamma(s - 2m, t - 2n)}{\Gamma(s, t)} \mathcal{M}(s - 2m, t - 2n)$$

The parity problem

Our $G(z, \bar{z}, \alpha, \bar{\alpha})$ cannot be written as $G(U, V, \alpha, \bar{\alpha})$!

$$V^{\pm} \sim V_{\mu} \pm \epsilon_{\mu\nu} V^{\nu} \subset s^{l_1}_{k_1} \times s^{l_2}_{k_2}, s^{l_1}_{k_1} \times \sigma_{k_2}, \sigma_{k_1} \times \sigma_{k_2}$$

$$\Rightarrow x^{\mu}_{12} x^{\nu}_{34} \epsilon_{\mu\nu} \subset G(z, \bar{z}, \alpha, \bar{\alpha}).$$

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Solve by defining two Mellin amplitudes.

$$G^{(s)}(z, \bar{z}, \alpha, \bar{\alpha}) = G^{(s,+)}(U, V, \alpha, \bar{\alpha}) + \frac{z - \bar{z}}{U} G^{(s,-)}(U, V, \alpha, \bar{\alpha})$$

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Considering chiral blocks $f_h(z) = z^h {}_2F_1(h, h; 2h; z)$,

$$G^{(s,+)} \supset f_h(z) f_{\bar{h}}(\bar{z}) + f_{\bar{h}}(z) f_h(\bar{z}) \Rightarrow \mathcal{M}^{2d}_{\Delta,\ell}$$

$$G^{(s,-)} \supset \frac{U}{z - \bar{z}} \left[f_h(z) f_{\bar{h}}(\bar{z}) - f_{\bar{h}}(z) f_h(\bar{z}) \right] \Rightarrow \mathcal{M}^{4d}_{\Delta+1,\ell-1}.$$

Consider $G^{I_1 I_2 I_3 I_4} = G^{(s)}\delta^{I_1 I_2}\delta^{I_3 I_4} + G^{(t)}\delta^{I_1 I_4}\delta^{I_2 I_3} + G^{(u)}\delta^{I_1 I_3}\delta^{I_2 I_4}$ in $\left\langle s_p^{I_1} s_p^{I_2} s_q^{I_3} s_q^{I_4} \right\rangle$ with $p \leq q$.

$$\mathcal{M}^{(s,\pm)}(s,t,\alpha,\bar{\alpha}) = \textit{known} + \sum_{j \leq i}^{p} [P_i(2\alpha - 1)P_j(2\bar{\alpha} - 1) \pm (i \leftrightarrow j)] \\ [A_{\pm}^{i,j} + (t+u)B_{\pm}^{i,j} + (t-u)C_{\pm}^{i,j}]$$

Survey of results

Consider $G^{I_1 I_2 I_3 I_4} = G^{(s)}\delta^{I_1 I_2}\delta^{I_3 I_4} + G^{(t)}\delta^{I_1 I_4}\delta^{I_2 I_3} + G^{(u)}\delta^{I_1 I_3}\delta^{I_2 I_4}$ in $\left\langle s_p^{I_1} s_p^{I_2} s_q^{I_3} s_q^{I_4} \right\rangle$ with $p \le q$.

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$$A_+^{i+1,i-1} = -\frac{pqi(i+1)(p+q)!}{4(2i+1)(p+i)!(q+i)!(p-i-1)!(q-i-1)!}.$$

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Results agree with [Giusto, Russo, Tyukov, Wen; 1905.12314] which took the form

$$G^{I_1 I_2 I_3 I_4}(z,\bar{z},\alpha,\bar{\alpha}) = \hat{G}^{I_1 I_2 I_3 I_4}(z,\bar{z},\alpha,\bar{\alpha}) + (1 - z\alpha)(1 - \bar{z}\bar{\alpha})H^{I_1 I_2 I_3 I_4}(U,V)$$
$$\widetilde{\mathcal{M}}^{(s)}(s,t,\sigma,\tau) = \sum_{0 \leq i+j \leq p-1} \frac{\sigma^j \tau^{p-i-j-1}}{[i!j!(p-i-j-1)!]^2(s+2i-2p+2)}.$$

Survey of results

Similar truncation for $\left\langle s_p^{I_1} s_p^{I_2} \sigma_q \sigma_q \right\rangle$ with degree 6 polynomials.

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Result in [Wen, Zhang; 2106.03499] gave even/odd Mellin amplitude using

$$H(U, V, \alpha, \bar{\alpha}) \mapsto H^{(+)}(U, V, \alpha, \bar{\alpha}) + (z - \bar{z}) H^{(-)}(U, V, \alpha, \bar{\alpha}).$$

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For non-truncated $\langle \sigma_p \sigma_p \sigma_q \sigma_q \rangle$ of degree 13, jump right to

$$\widetilde{\mathcal{M}}^{(-)}(s, t, \alpha, \bar{\alpha}) = \frac{2p^2 q^2 (p^2 + q^2 - 2)(\alpha - \bar{\alpha})}{(p+1)!(q+1)!} \sum_{0 \leq i+j \leq p-2} \frac{\sigma^i \tau^j}{(i!j!)^2} \times \frac{(p-i-j-1)_{i+j}(q-i-j-1)_{i+j}}{(s-2i-2j-2)(t-p-q+2j+2)(\tilde{u}-p-q+2i+2)}$$

for $\tilde{u} = u - 4$ [CB, Pitombo; 2408.17420] .

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for $\tilde{u} = u - 4$ [CB, Pitombo; 2408.17420] . Also

$$\widetilde{\mathcal{M}}^{(+)}(s, t, \alpha, \bar{\alpha}) = \sum_{0 \leq i+j \leq p-1} \frac{\sigma^i \tau^j}{(i!j!)^2} \left[\sum_{m=-1}^{1} \left(\frac{d_s(m)}{s - s_m} + \frac{d_t(m)}{t - t_m} + \frac{d_u(m)}{\tilde{u} - u_m} \right) + \sum_{n=-1}^{1} \frac{d_{tu}(n)}{(t - t'_n)(\tilde{u} - u'_n)} + \sum_{m=0}^{2} \sum_{n=m-1}^{1} \frac{1}{s - s''_m} \left(\frac{d_{st}(m,n)}{t - t''_n} + \frac{d_{su}(m,n)}{\tilde{u} - u''_n} \right) \right]$$

for $\tilde{u} = u - 2$ whose flat limit is $\frac{1}{stu}$ times a function of (s, t, p, q).

Missing correlators

$\left\langle s^{I_1}_{k_1} s^{I_2}_{k_2} V^+_{k_3} V^+_{k_4} \right\rangle$ $\left\langle \sigma_{k_1} \sigma_{k_2} V^+_{k_3} V^+_{k_4} \right\rangle$ $\left\langle V^+_{k_1} V^+_{k_2} V^-_{k_3} V^-_{k_4} \right\rangle$ $\left\langle V^+_{k_1} V^+_{k_2} V^+_{k_3} V^+_{k_4} \right\rangle$

Do they need SUGRA work beyond [Arutyunov, Pankiewicz, Theisen; hep-th/0007601] ?

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Chiral OPE becomes applicable in twisted configuration if superconformal symmetry is obeyed [Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees; 1312.5344] .

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Algorithm for the spinning case

External scalars have ${}_2F_1$ blocks with $h_{12} = \bar{h}_{12}$. If not, use

$$\left(z \frac{\partial}{\partial z} z \right)^n \left[z^{a-1} {}_2F_1(a, b; c; z) \right] = (a)_n z^{a+n-1} {}_2F_1(a+n, b; c; z).$$

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This is 100 operators because $m \equiv h_{12} - \bar{h}_{12}$ and $n \equiv h_{34} - \bar{h}_{34}$ are both in $\{-2, -1, 0, 1, 2\}$.

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$\sigma_{k_1} \times V^-_{k_2}$	1
$\sigma_{k_1} \times V^+_{k_2}$	-1
$V^+_{k_1} \times V^-_{k_2}$	2
$V^-_{k_1} \times V^+_{k_2}$	-2

Future directions

- The same methods should be applied to $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$.
- KK-modes allow a broad exploration of AdS string amplitudes which has been started in [Chester, Zhong; 2412.06429] .
- Even/odd tree amplitudes for $\left\langle s_1^{l_1} s_1^{l_2} V_k^{\pm} V_k^{\pm} \right\rangle$ determine the one-loop correction to $\left\langle s_1^{l_1} s_1^{l_2} s_1^{l_3} s_1^{l_4} \right\rangle$ [Aharony, Alday, Bissi, Perlmutter; 1612.03891] .
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