How far can the Coulomb gas go?

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DESY seminar

* Oxford University

Related: 2xxx.xxxxx with A. Antunes

Review of minimal models

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W-minimal models

Simplest W-algebra in [Zamolodchikov; 86] is part of an ADE family. Can define using coset construction $W[\hat{\mathfrak{g}}] = \frac{\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_1}{\hat{\mathfrak{g}}_{k+1}}$, Casimir construction

$$T(z) \equiv W^{(2)}(z) = \delta_{ab}(J^a J^b)(z), \quad W^{(3)}(z) = d_{abc}(J^a(J^b J^c))(z), \quad \ldots$$

or quantum Drinfeld Sokolov reduction.

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For charges solving $\alpha_\pm^2 - \alpha_0\alpha_\pm = \frac{1}{2}$, we had $c = 1 - 12\alpha_0^2$ and

$$h_{(\lambda;\lambda')} = (\beta_{(\lambda;\lambda')}, \beta_{(\lambda;\lambda')} - \sqrt{2}\alpha_0), \quad \beta_{(\lambda;\lambda')} = -\left[\alpha_+\lambda + \alpha_-\lambda'\right]\frac{1}{\sqrt{2}}$$

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For charges solving $\alpha_\pm^2 - \alpha_0\alpha_\pm = \frac{1}{2}$, now $c = rank(\mathfrak{g}) - 24\alpha_0^2\rho^2$ and

$$h_{(\lambda;\lambda')} = (\beta_{(\lambda;\lambda')}, \beta_{(\lambda;\lambda')} - 2\alpha_0\rho), \quad \beta_{(\lambda;\lambda')} = -\sum_i [\alpha_+\lambda_i + \alpha_-\lambda'_i]\,\omega_i$$

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Multi-component Coulomb gas

Free field realization of Virasoro minimal models looked like

$$S = \frac{1}{16\pi} \int \mathrm{d}^2 z \, \sqrt{g} \left[(\partial_\alpha X)^2 + 2\sqrt{2} i \alpha_0 R X + V_{\sqrt{2}\alpha_+} + V_{\sqrt{2}\alpha_-} \right]$$

with the correspondence

$$\phi_{(\lambda;\lambda')}(z,\bar{z}) \leftrightarrow N^{-1}_{(\lambda;\lambda')} V_{\beta_{(\lambda;\lambda')}}(z,\bar{z}) \equiv N^{-1}_{(\lambda;\lambda')} e^{i(\beta_{(\lambda;\lambda')}, X)(z,\bar{z})}.$$

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Note that $V_{\beta_{(\lambda;\lambda')}}$ and $V_{2\alpha_0\rho-\beta^*_{(\lambda;\lambda')}}$ map to same primary with inverse normalizations. In order to compute $C_{1,2,3}$,

$$\begin{aligned}
N_1 N_2 N_3^{-1} C^3_{1,2} &= \left\langle V_{\beta_1} V_{\beta_2} V_{2\alpha_0\rho-\beta_3^*} \dots \right\rangle \quad & N_1^2 &= \langle V_1 V_1 V_{2\alpha_0\rho} \dots \rangle \\
N_2 N_3 N_1^{-1} C^1_{2,3} &= \left\langle V_{\beta_2} V_{\beta_3} V_{2\alpha_0\rho-\beta_1^*} \dots \right\rangle \quad & N_2^2 &= \langle V_2 V_2 V_{2\alpha_0\rho} \dots \rangle \\
N_3 N_1 N_2^{-1} C^2_{3,1} &= \left\langle V_{\beta_3} V_{\beta_1} V_{2\alpha_0\rho-\beta_2^*} \dots \right\rangle \quad & N_3^2 &= \langle V_3 V_3 V_{2\alpha_0\rho} \dots \rangle
\end{aligned}$$

Useful integrals

Building blocks in terms of $\gamma(z) \equiv \Gamma(z)/\Gamma(1-z)$ are

$$K_1(\alpha,\beta) = \int d^2z\,|z|^{2\alpha}|z-1|^{2\beta} = \pi\frac{\gamma(\alpha+1)\gamma(\beta+1)}{\gamma(\alpha+\beta+2)}$$

$$K_2(\alpha,\beta) = \int d^2z_1 d^2z_2\,(|z_1||z_1-1||z_2||z_2-1|)^{2\alpha}|z_{12}|^{4\beta}$$

$$= 2\pi^2\frac{\gamma(2\beta)}{\gamma(\beta)}\frac{\gamma(\alpha+1)^2}{\gamma(2\alpha+\beta+2)}\frac{\gamma(\alpha+\beta+1)^2}{\gamma(2\alpha+2\beta+2)}$$

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$$N_{(0,0;1,1)}\, C^{(0,0;1,1)}_{(0,0;1,1)(0,0;1,1)} \cdots \leq 2, \quad N^2_{(0,0;1,1)} \cdots > 2$$

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$$N^2_{(0,0;0,1)},\ N_{(0,0;1,1)}\, C^{(0,0;1,1)}_{(0,0;0,1)(0,0;1,0)},\ N^2_{(0,0;0,1)} N^{-1}_{(0,0;1,1)}\, C^{(0,0;1,1)}_{(0,0;0,1)(0,0;1,0)} \cdots \leq 2!$$

Limits to this method

Despite some generalizations ([Fukuda, Hosomichi; hep-th/0105217] [Fateev, Litvinov; 0709.3806]), three basic families are out of reach.

1. Weights too large and too far from extremality. E.g. if $C^{\beta}_{\beta,\beta}$ is non-trivial, $C^{N\beta}_{N\beta,N\beta}$ requires N times as many of each screening charge.
2. Norm is already too complicated for all fundamental representations. Happens for $\mathfrak{e}_6$ where simplest norm $N^2_{(\mathbf{0};1,0,0,0,0,0)} = N^2_{(\mathbf{0};0,0,0,0,1,0)}$ requires multiplicities of 2, 3, 4, 3, 2, 2.
3. Norms which are not too complicated might form a closed subsector. Happens for tensor representations of $\mathfrak{d}_n$.

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These are conformal integrals ($\sum_i \Delta_i = d$). After using

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$$= 10\pi^4 (2\alpha_-^2 - 1)^{-4} + \dots \quad (1 \ \mathit{hour})$$

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Screening 4pt functions instead

With 4pt functions of the form

$$\left\langle \phi_{(\lambda_1;\lambda_1')}(0)\phi_{(\lambda_1;\lambda_1')}(z,\bar{z})\phi_{(\lambda_2;\lambda_2')}(1)\phi_{(\lambda_2;\lambda_2')}(\infty) \right\rangle = \sum_{j=1}^{M} X_j \left| I_j(z) \right|^2$$

$$I_j(z) = N_j z^{h_j - 2h_{(\lambda_1;\lambda_1')}} \left[1 + O(z)\right]$$

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we can use [Dotsenko, Fateev; 84] [Dotsenko, Fateev; 85]. If we can compute all N_j and express

$$I_j(z) = \sum_k F_{jk}\tilde{I}_k(z), \quad \tilde{I}_k(1-z) = \tilde{N}_k z^{h_k - h_{(\lambda_1;\lambda_1')} - h_{(\lambda_2;\lambda_2')}}\left[1 + O(z)\right],$$

1. Killing $(1-z)^{h_j}(1-\bar{z})^{h_k}$ terms fixes $\frac{X_j}{X_k} = \frac{F^*_{kk}(F^{-1})_{kj}}{F^*_{jk}(F^{-1})_{kk}}$.
2. Norms allow us to write $\frac{C^j_{(\lambda_1;\lambda_1')(\lambda_1;\lambda_1')}C^j_{(\lambda_2;\lambda_2')(\lambda_2;\lambda_2')}}{C^k_{(\lambda_1;\lambda_1')(\lambda_1;\lambda_1')}C^k_{(\lambda_2;\lambda_2')(\lambda_2;\lambda_2')}} = \frac{N_j^2}{N_k^2}\frac{X_j}{X_k}$.
3. Identity term $C^{(\mathbf{0};\mathbf{0})}_{(\lambda_1;\lambda_1')(\lambda_1;\lambda_1')}C^{(\mathbf{0};\mathbf{0})}_{(\lambda_2;\lambda_2')(\lambda_2;\lambda_2')} = 1$ completes solution.

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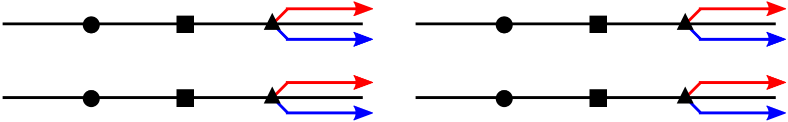

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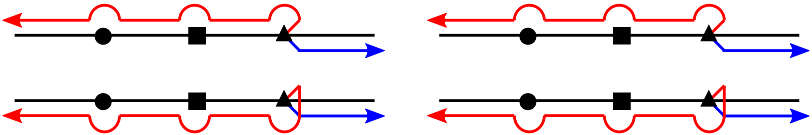

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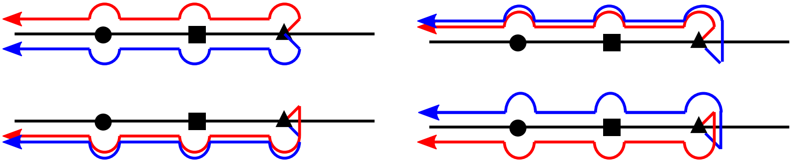

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From Virasoro to W-algebras

Complete Virasoro solution allows us to go back and compute

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Use `MB.m` to extract most singular term in $\epsilon = 2\alpha_-^2 - 1$.

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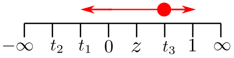

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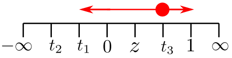

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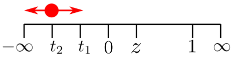

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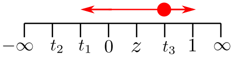

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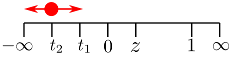

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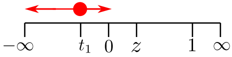

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Contour validation

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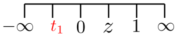

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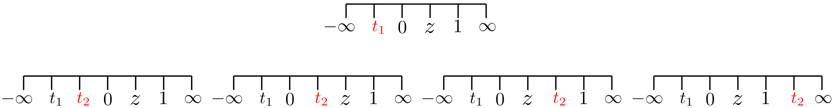
The integrand $[t_1(t_1 - z)(t_3 - 1)t_{12}t_{13}t_{23}]^{-2\alpha_-^2}$ has a problem with $\int_{-\infty}^{0} \mathrm{d}t_1 \int_{t_1}^{\infty} \mathrm{d}t_2 \int_{t_1}^{1} \mathrm{d}t_3$.



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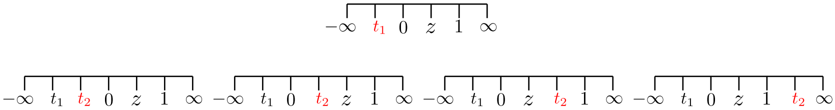

Bad Bad Bad Good

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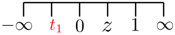

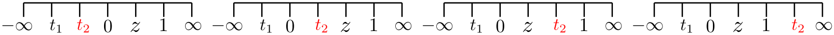

Bad Bad Bad Good

t_1 interval	t_2 interval
$(-\infty, 0)$	$(-\infty, t_1)$, $(t_1, 0)$, $(0, 1)$, $(1, \infty)$
$(0, z)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(z, 1)$	$(-\infty, 0)$, $(0, t_1)$, $(t_1, 1)$, $(1, \infty)$
$(1, \infty)$	$(-\infty, 0)$, $(0, 1)$, $(1, t_1)$, (t_1, ∞)

Returning to the 16 contour problem, 8 relations are easy.

$$\int_{-\infty}^{0} \mathrm{d}t_1 \left[\int_{-\infty}^{t_1} + e^{\pm 2\pi i \alpha_-^2} \int_{t_1}^{0} + e^{\pm 4\pi i \alpha_-^2} \int_{0}^{1} + e^{\pm 6\pi i \alpha_-^2} \int_{1}^{\infty} \right] \mathrm{d}t_2 = 0$$

Finishing the example

Trying $[t_1(t_1 - z)(t_1 - 1)t_2(t_2 - 1)t_{12}]^{-2\alpha_-^2}$ with t_2 first,

$$\int_0^1 \mathrm{d}t_2 \left[\int_{-\infty}^0 \mathrm{d}t_1 + \dots \right] = 0.$$

Finishing the example

Trying $[t_1(t_1 - z)(t_1 - 1)t_2(t_2 - 1)t_{12}]^{-2\alpha_-^2}$ with t_2 first,

$$\int_0^1 dt_2 \left[\int_{-\infty}^0 dt_1 + e^{\pm 2\pi i \alpha_-^2} \int_0^{t_2?z?} dt_1 + \dots \right] = 0.$$

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Split $(0,1) = (0,z) \cup (z,1)$ even though $t_2 \to z$ is non-singular.

$$\int_0^1 \mathrm{d}t_2 \int_{-\infty}^0 \mathrm{d}t_1 + \int_0^z \mathrm{d}t_2 \left[e^{\pm 2\pi i \alpha_-^2} \int_0^{t_2} + e^{\pm 4\pi i \alpha_-^2} \int_{t_2}^z + e^{\pm 6\pi i \alpha_-^2} \int_z^1 \right] \mathrm{d}t_1 +$$

$$\int_z^1 \mathrm{d}t_2 \left[e^{\pm 2\pi i \alpha_-^2} \int_0^z + e^{\pm 4\pi i \alpha_-^2} \int_z^{t_2} + e^{\pm 6\pi i \alpha_-^2} \int_{t_2}^1 \right] \mathrm{d}t_1 + e^{\pm 8\pi i \alpha_-^2} \int_0^1 \mathrm{d}t_2 \int_1^\infty \mathrm{d}t_1 = 0$$

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$$\int_z^1 \mathrm{d}t_2 \left[e^{\pm 2\pi i\alpha_-^2}\int_0^{z} + e^{\pm 4\pi i\alpha_-^2}\int_{z}^{t_2} + e^{\pm 6\pi i\alpha_-^2}\int_{t_2}^1\right]\mathrm{d}t_1 + e^{\pm 8\pi i\alpha_-^2}\int_0^1 \mathrm{d}t_2\int_1^\infty \mathrm{d}t_1 = 0$$

Change of order makes this

$$\int_{-\infty}^0 \mathrm{d}t_1 \int_0^1 \mathrm{d}t_2 + e^{\pm 8\pi i\alpha_-^2}\int_1^\infty \mathrm{d}t_1\int_0^1 \mathrm{d}t_2 +$$
$$e^{\pm 2\pi i\alpha_-^2}\int_0^z \mathrm{d}t_1\int_{t_1}^z \mathrm{d}t_2 + e^{\pm 4\pi i\alpha_-^2}\int_0^z \mathrm{d}t_1\int_0^{t_1} \mathrm{d}t_2 + e^{\pm 6\pi i\alpha_-^2}\int_z^1 \mathrm{d}t_1\int_0^z \mathrm{d}t_2 +$$
$$e^{\pm 2\pi i\alpha_-^2}\int_0^z \mathrm{d}t_1\int_z^1 \mathrm{d}t_2 + e^{\pm 4\pi i\alpha_-^2}\int_z^1 \mathrm{d}t_1\int_{t_1}^1 \mathrm{d}t_2 + e^{\pm 6\pi i\alpha_-^2}\int_z^1 \mathrm{d}t_1\int_z^{t_1} \mathrm{d}t_2 = 0.$$

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$$\int_0^1 dt_2 \int_{-\infty}^0 dt_1 + \int_0^z dt_2 \left[e^{\pm 2\pi i \alpha_-^2} \int_0^{t_2} + e^{\pm 4\pi i \alpha_-^2} \int_{t_2}^z + e^{\pm 6\pi i \alpha_-^2} \int_z^1 \right] dt_1 +$$

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$$e^{\pm 2\pi i\alpha_-^2} \int_0^z \mathrm{d}t_1 \int_{t_1}^1 \mathrm{d}t_2 + e^{\pm 4\pi i\alpha_-^2} \int_0^z \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 + e^{\pm 6\pi i\alpha_-^2} \int_z^1 \mathrm{d}t_1 \int_0^{t1} \mathrm{d}t_2 +$$

$$e^{\pm 4\pi i\alpha_-^2} \int_z^1 \mathrm{d}t_1 \int_{t_1}^1 \mathrm{d}t_2 \qquad = 0.$$

Symmetry relations

The ways to reorder n integrals are elements of $G = S_n$.
Screening charge assignment of $V_{\alpha-\alpha_1}^{n_1} \dots V_{\alpha-\alpha_{rank(\mathfrak{g})}}^{n_{rank(\mathfrak{g})}}$ has symmetry group $H = S_{n_1} \times \cdots \times S_{n_{rank(\mathfrak{g})}}$.

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$$\left\langle \phi_{(\mathbf{0};0,1,0)}(0)\phi_{(\mathbf{0};0,1,0)}(z)\phi_{(\mathbf{0};0,1,0)}(1)\phi_{(\mathbf{0};0,1,0)}(\infty)\right\rangle = \oint \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4$$

$$[t_1 t_2 (t_1 - z)(t_2 - z)(t_1 - 1)(t_2 - 1)]^{-2\alpha_-^2} t_{12}^{4\alpha_-^2} [t_{13} t_{14} t_{23} t_{24}]^{-2\alpha_-^2}$$

Algorithm generates 180 contours, 90 after symmetries, 10 after easy relations, 3 after hard relations.

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- $\int_0^z \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_{t_2}^{t_1} \mathrm{d}t_3 \int_{t_2}^{t_1} \mathrm{d}t_4 \Rightarrow \phi_{(\mathbf{0};0,0,0)}$
- $\int_0^z \mathrm{d}t_1 \int_1^{\infty} \mathrm{d}t_2 \int_{t_1}^{t_2} \mathrm{d}t_3 \int_{t_1}^{t_2} \mathrm{d}t_4 \Rightarrow \phi_{(\mathbf{0};1,0,1)}$
- $\int_1^{\infty} \mathrm{d}t_1 \int_1^{t_1} \mathrm{d}t_2 \int_{t_2}^{t_1} \mathrm{d}t_3 \int_{t_2}^{t_1} \mathrm{d}t_4 \Rightarrow \phi_{(\mathbf{0};0,2,0)}$

Good and bad mixed correlators

- We can extract C_{HHL} from $\langle HHHH \rangle$ but $\langle HHLL \rangle$ is better.
- To get different powers of z, at least one $V_{\alpha_{-\alpha_i}}(t)$ must be integrated from 0 to z.
- Consider $\left\langle \phi_{(\mathbf{0};\lambda_1)}(0)\phi_{(\mathbf{0};\lambda_2)}(z)\phi_{(\mathbf{0};\lambda_3)}(1)\phi_{(\mathbf{0};\lambda_4)}(\infty) \right\rangle$. Left end on $0 \Rightarrow (\lambda_1, \alpha_i) \neq 0$. Right end on $z \Rightarrow (\lambda_2, \alpha_i) \neq 0$.

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We still need the matrix elements $\alpha_{km}^{(m)}$, to use formula (2.13) and find the structure constants X_k of the four-point correlation functions. One can check that the technique, which is used above, does not lead to easy calculations in the case of the matrix elements α_{km}. So, we use an alternative way.

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

$$I_*(y) = \int_1^{\infty} \mathrm{d}t_1 \cdots \int_1^{t_{m-1}} \mathrm{d}t_m \int_0^{1-y} \mathrm{d}t_{m+1} \cdots \int_0^{t_{n-1}} \mathrm{d}t_n$$

$$\prod_{i=1}^{m} t_i^a (t_i - 1 + y)^b (t_i - 1)^c \prod_{i=m+1}^{n} t_i^a (1 - y - t_i)^b (1 - t_i)^c \prod_{i<j} t_{ij}^{2q}$$

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With $t_1, \dots t_m$, perform $t_i \mapsto 1 + t_i y$ on last k.
With $t_{m+1}, \dots t_n$, perform $t_i \mapsto 1 - t_i y$ on first l.

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$$\prod_{i=1}^{m} t_i^a (t_i - 1 + y)^b (t_i - 1)^c \prod_{i=m+1}^{n} t_i^a (1 - y - t_i)^b (1 - t_i)^c \prod_{i<j} t_{ij}^{2q}$$

With $t_1, \ldots t_m$, perform $t_i \mapsto 1 + t_i y$ on all of them.
With $t_{m+1}, \ldots t_n$, perform $t_i \mapsto 1 - t_i y$ on all of them.

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

$$I_*(y) = y^{n[(n-1)q+b+c+1]} \int_0^\infty \mathrm{d}t_1 \cdots \int_0^{t_{m-1}} \mathrm{d}t_m \int_1^{1/y} \mathrm{d}t_{m+1} \cdots \int_{t_{n-1}}^{1/y} \mathrm{d}t_n$$

$$\prod_{i=1}^{m} (1 + t_i y)^a (t_i + 1)^b t_i^c \prod_{i=m+1}^{n} (1 - t_i y)^a (t_i - 1)^b t_i^c$$

$$\prod_{\substack{i,j=1 \\ i<j}}^{m} t_{ij}^{2q} \prod_{\substack{i,j=m+1 \\ i<j}}^{n} t_{ji}^{2q} \prod_{i=1}^{m} \prod_{j=m+1}^{n} (t_i + t_j)^{2q}$$

With $t_1, \ldots t_m$, perform $t_i \mapsto 1 + t_i y$ on all of them.
With $t_{m+1}, \ldots t_n$, perform $t_i \mapsto 1 - t_i y$ on all of them.

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

$$I_*(y) \approx y^{n[(n-1)q+b+c+1]} \int_0^\infty \mathrm{d}t_1 \cdots \int_0^{t_{m-1}} \mathrm{d}t_m \int_1^\infty \mathrm{d}t_{m+1} \cdots \int_{t_{n-1}}^\infty \mathrm{d}t_n$$

$$\prod_{i=1}^{m} (t_i + 1)^b t_i^c \prod_{i=m+1}^{n} (t_i - 1)^b t_i^c$$

$$\prod_{\substack{i,j=1 \\ i<j}}^{m} t_{ij}^{2q} \prod_{\substack{i,j=m+1 \\ i<j}}^{n} t_{ji}^{2q} \prod_{i=1}^{m} \prod_{j=m+1}^{n} (t_i + t_j)^{2q}$$

With $t_1, \ldots t_m$, perform $t_i \mapsto 1 + t_i y$ on all of them.
With $t_{m+1}, \ldots t_n$, perform $t_i \mapsto 1 - t_i y$ on all of them.
We can now zoom in on a single power of y.

The other crossing matrix derivation

Virasoro basis element with $y = 1 - z$.

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$$\prod_{i=1}^m (t_i+1)^b t_i^c \prod_{i=m+1}^n (t_i-1)^b t_i^c$$

$$\prod_{\substack{i,j=1\\ i<j}}^m t_{ij}^{2q} \prod_{\substack{i,j=m+1\\ i<j}}^n t_{ji}^{2q} \prod_{i=1}^m \prod_{j=m+1}^n (t_i+t_j)^{2q}$$

With $t_1, \ldots t_m$, perform $t_i \mapsto 1 + t_i y$ on all of them.
With $t_{m+1}, \ldots t_n$, perform $t_i \mapsto 1 - t_i y$ on all of them.
We can now zoom in on a single power of y.
Does not increase difficulty of $\epsilon = 2\alpha_-^2 - 1$ expansion.

A bad mixed correlator

Two exchanged blocks but only one possible local exponent.

$$\left\langle \phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};0,1)}\phi_{(\mathbf{0};1,0)}\phi_{(\mathbf{0};0,1)} \right\rangle = \oint \mathrm{d}t_1 \mathrm{d}t_2 \left[t_1(t_2 - z)(t_1 - 1)t_{12}\right]^{-2\alpha_-^2}$$

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Monodromy relations narrow the basis down to two integrals.

$$I_1(z) = \int_{-\infty}^{0} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots, \quad I_2(z) = \int_{1}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{z} \mathrm{d}t_2 \ldots$$

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Two exchanged blocks but only one possible local exponent.

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$$I_1(z) = \int_{-\infty}^{0} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \dots, \quad I_2(z) = \int_{1}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{z} \mathrm{d}t_2 \dots$$

Crossing symmetry only fixes $X_2 = X_1$.

$$\frac{G(z,\bar{z})}{|z|^{\frac{32}{3}\alpha_-^2 - 4}} = X_0 [I_1(z) I_2(z)^* + I_1(z)^* I_2(z)] + X_1 |I_1(z)|^2 + X_2 |I_2(z)|^2$$

Exchanges of $\phi_{(\mathbf{0};1,1)}$ and $\phi_{(\mathbf{0};0,0)}$ correspond to z^0 and $z^{2-6\alpha_-^2}$.

A bad mixed correlator

For small z in s-channel and small y in t-channel,

$$I_1(z) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(z)], \quad I_2(z) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(z)].$$

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$$I_1(y) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(y)]$$

$$I_2(y) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(y)] + \frac{1}{(2\alpha_-^2 - 1)^2} y^{2-6\alpha_-^2}[1 + O(y)].$$

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For small z in s-channel and small y in t-channel,

$$I_1(z) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(z)], \quad I_2(z) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(z)].$$

$$I_1(y) = \frac{1}{(2\alpha_-^2 - 1)^2}[1 + O(y)]$$

$$I_2(y) = -\frac{1}{3(2\alpha_-^2 - 1)^2}[1 + O(y)] + \frac{1}{(2\alpha_-^2 - 1)^2} y^{2-6\alpha_-^2}[1 + O(y)].$$

Trivial monodromy in y now fixes $X_0 = \frac{1}{3}X_2$.

$$\frac{G(z,\bar{z})}{|z|^{\frac{32}{3}\alpha_-^2 - 4}} = (2\alpha_-^2 - 1)^4 \left[|I_1(z)|^2 + |I_2(z)|^2 + \frac{2}{3} Re I_1(z) I_2(z)^* \right]$$

A bad mixed correlator

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$$\supset |z|^{16\alpha_-^2 - 4} \left[\frac{1}{4} + \frac{1}{4} + \frac{2}{3}\frac{1}{4} \right], \quad z \to \infty?$$

Expanding in the u-channel

First step in verifying $C^{(\mathbf{0};0,2)}_{(\mathbf{0};0,1)(\mathbf{0};0,1)} = \sqrt{\frac{2}{3}}$ comes from $t_i \mapsto (zt_i)^{-1}$.

$$I_1\left(\frac{1}{z}\right) = z^{8\alpha_-^2-2}\int_{-\infty}^{0}\frac{\mathrm{d}t_1}{t_1^2}\int_{t_1}^{0}\frac{\mathrm{d}t_2}{t_2^2}\left[\frac{(1-t_2)(t_2-t_1)(1-t_1 z)}{(-t_3)^3(-t_2)^2}\right]^{-2\alpha_-^2}$$

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For the next integral, try $t_i \mapsto t_i/z$.

$$I_2\left(\frac{1}{z}\right) = z^{-2}\int_{z}^{\infty}\mathrm{d}t_1\int_{-\infty}^{1}\mathrm{d}t_2\ldots$$

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Write integrand as $\left(\frac{t_{12}}{z}\right)^{-2\alpha_-^2}\left[\frac{t_1}{z}\frac{1-t_2}{z}\frac{t_1-z}{z}\right]^{-2\alpha_-^2}$ or $e^{-2\pi i\alpha_-^2}\left(\frac{t_{21}}{z}\right)^{-2\alpha_-^2}\left[\frac{t_1}{z}\frac{1-t_2}{z}\frac{t_1-z}{z}\right]^{-2\alpha_-^2}$ to get

$$I_2\left(\frac{1}{z}\right) = \frac{z^{8\alpha_-^2-2}}{(2\alpha_-^2-1)^2}\left(1-2+\frac{1}{2}\right)[1+O(z)].$$

The real test

Extending preturbative results of [Dotsenko, Nguyen, Santachiara; hep-th/0104197] requires $C^{(\mathbf{0};0,1,0,0,0,0)}_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)}$ in $W[\hat{\mathfrak{d}}_6]$. Start with $W[\hat{\mathfrak{d}}_4]$.

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- $\langle V(0)S(z)V(1)S(\infty)\rangle$ has 24 integrals and 22 relations.
- Exchanged $\phi_{(\mathbf{0};0,0,1,0)}$ is only in $I_1(z)$.
- Exchanged $\phi_{(\mathbf{0};1,0,0,1)}$ lives in $I_1(z)$ and image under $z \leftrightarrow 1-z$.
- $\langle V(0)V(z)S(1)S(\infty)\rangle$ has 648 integrals but only 632 relations.

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One integral has $z^{6-16\alpha_-^2}$ for $\phi_{(\mathbf{0};0,0,0,0)}$.

$$I_1(z) = \int_0^z \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_{t_2}^{t_1} \mathrm{d}t_3 \int_{t_2}^{t_3} \mathrm{d}t_4 \int_{t_4}^{t_3} \mathrm{d}t_5 \int_{t_4}^{t_3} \mathrm{d}t_6 \ldots$$

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Eight have $z^{1-4\alpha_-^2}$ for $\phi_{(\mathbf{0};0,1,0,0)}$, four can help us cancel $y^0\bar{y}^{3-8\alpha_-^2}$.

$$I_2(z) = \int_{-\infty}^0 \mathrm{d}t_1 \int_0^z \mathrm{d}t_2 \int_{-\infty}^{t_1} \mathrm{d}t_3 \int_{t_1}^{t_2} \mathrm{d}t_4 \int_{t_3}^{t_4} \mathrm{d}t_5 \int_{t_3}^{t_4} \mathrm{d}t_6 \dots ?$$

Others lead to “weird” numbers including $(2\alpha_-^2 - 1)^{-4}$ or $(2\alpha_-^2 - 1)^{-5}$.

The real test

Extending preturbative results of [Dotsenko, Nguyen, Santachiara; hep-th/0104197] requires $C^{(\mathbf{0};0,1,0,0,0,0)}_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)}$ in $W[\hat{\mathfrak{d}}_6]$. Start with $W[\hat{\mathfrak{d}}_4]$.

- $\langle V(0)S(z)V(1)S(\infty)\rangle$ has 1536 integrals and 1528 relations.
- Exchanged $\phi_{(\mathbf{0};0,0,0,0,1,0)}$ is only in $I_1(z)$.
- Exchanged $\phi_{(\mathbf{0};1,0,0,0,0,1)}$ lives in $I_1(z)$ and image under $z \leftrightarrow 1-z$.
- $\langle V(0)V(z)S(1)S(\infty)\rangle$ has 40992 integrals, not enough relations.

One integral has $z^{10-24\alpha_-^2}$ for $\phi_{(\mathbf{0};0,0,0,0,0,0)}$.

$$I_1(z) = \int_0^z dt_1 \int_0^{t_1} dt_2 \int_{t_2}^{t_1} dt_3 \int_{t_2}^{t_3} dt_4 \int_{t_4}^{t_3} dt_5 \int_{t_4}^{t_5} dt_6 \int_{t_6}^{t_5} dt_7 \int_{t_6}^{t_7} dt_8 \int_{t_8}^{t_7} dt_9 \int_{t_8}^{t_7} dt_{10} .$$

Many have $z^{1-4\alpha_-^2}$ for $\phi_{(\mathbf{0};0,1,0,0,0,0)}$, fewer can help us cancel $y^0\bar{y}^{5-12\alpha_-^2}$.

$$I_2(z) = \int_{-\infty}^0 dt_1 \int_0^z dt_2 \int_{-\infty}^{t_1} dt_3 \int_{t_1}^{t_2} dt_4 \int_{-\infty}^{t_3} dt_5 \int_{t_3}^{t_4} dt_6 \int_{-\infty}^{t_5} dt_7 \int_{t_5}^{t_6} dt_8 \int_{t_7}^{t_8} dt_9 \int_{t_7}^{t_8} d$$

Others lead to "weird" numbers including $(2\alpha_-^2 - 1)^{-8}$ or $(2\alpha_-^2 - 1)^{-9}$.

The real test

To reproduce $C^{(\mathbf{0};0,0,1,0)}_{(\mathbf{0};1,0,0,0)(\mathbf{0};0,0,0,1)} = \sqrt{2}$ in $W[\hat{\mathfrak{d}}_4]$,

$$N_1^{-2}|I_1(z)|^2 + XN_2^{-2}|I_2(z)|^2 \Rightarrow X = \frac{7}{4}.$$

Notice that $X = C^{(\mathbf{0};0,1,0,0)}_{(\mathbf{0};1,0,0,0)(\mathbf{0};1,0,0,0)} C^{(\mathbf{0};0,1,0,0)}_{(\mathbf{0};0,0,0,1)(\mathbf{0};0,0,0,1)}$!

The real test

To reproduce $C_{(\mathbf{0};1,0,0,0)(\mathbf{0};0,0,0,1)}^{(\mathbf{0};0,0,1,0)} = \sqrt{2}$ in $W[\hat{\mathfrak{d}}_4]$,

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To reproduce $C_{(\mathbf{0};1,0,0,0,0,0)(\mathbf{0};0,0,0,0,0,1)}^{(\mathbf{0};0,0,0,0,1,0)} = \sqrt{3}$ in $W[\hat{\mathfrak{d}}_6]$,

$$N_1^{-2}|I_1(z)|^2 + YN_2^{-2}|I_2(z)|^2 \Rightarrow Y = 11.$$

If Y is similarly special, $C_{(\mathbf{0};0,0,0,0,0,1)(\mathbf{0};0,0,0,0,0,1)}^{(\mathbf{0};0,1,0,0,0,0)} = \frac{11}{10}\sqrt{\frac{33}{2}}$.

The real test

To reproduce $C^{(\mathbf{0};0,0,1,0)}_{(\mathbf{0};1,0,0,0)(\mathbf{0};0,0,0,1)} = \sqrt{2}$ in $W[\hat{\mathfrak{d}}_4]$,

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To reproduce $C^{(\mathbf{0};0,0,0,0,1,0)}_{(\mathbf{0};1,0,0,0,0,0)(\mathbf{0};0,0,0,0,0,1)} = \sqrt{3}$ in $W[\hat{\mathfrak{d}}_6]$,

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- The "Coulomb gas method" is really three.
- All still needed pending a breakthrough in special functions.
- Most parts are now algorithmic but some puzzles remain.