# Bootstrapping some continuous families of conformal field theories

Connor Behan

Yang Institute for Theoretical Physics
PhD Defence

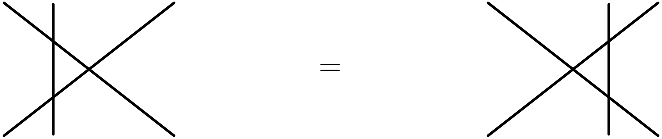
2019-08-09

# Outline

1. Review of bootstrap methods
2. Conformal field theories in non-integer dimension
3. Duality / bootstrap for the long-range (nonlocal) Ising model
4. "Nicer" continuous families

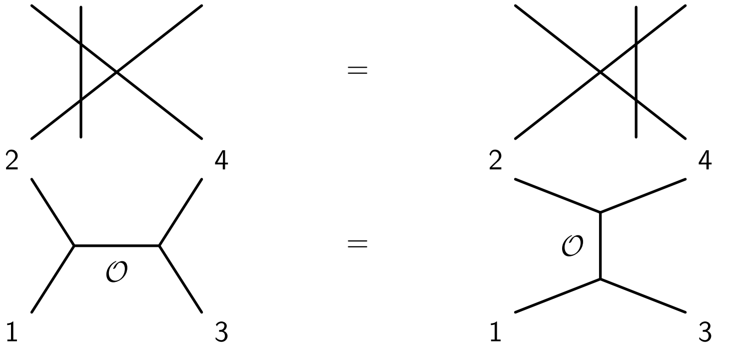
# Outline

1. Review of bootstrap methods
2. Conformal field theories in non-integer dimension
3. Duality / bootstrap for the long-range (nonlocal) Ising model
4. "Nicer" continuous families



# Outline

1. Review of bootstrap methods
2. Conformal field theories in non-integer dimension
3. Duality / bootstrap for the long-range (nonlocal) Ising model
4. “Nicer” continuous families



# QFT and critical phenomena

$$S = \int_{\mathbb{R}^4} \frac{1}{2}(\partial\phi)^2 + a(T - T_{\mathrm{c}})\phi^2 + \frac{g_0}{4!}\phi^4 dx$$

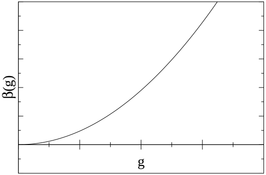
# QFT and critical phenomena

$$S = \int_{\mathbb{R}^4} \frac{1}{2}(\partial\phi)^2 + \frac{g_0}{4!}\phi^4 dx$$

# QFT and critical phenomena

$$S = \int_{\mathbb{R}^4} \frac{1}{2}(\partial\phi)^2 + \frac{g_0}{4!}\phi^4 dx$$

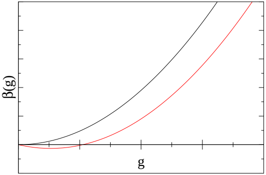
$$\beta(g) \equiv \mu\frac{dg}{d\mu} = \frac{3g^2}{(4\pi)^2}$$



# QFT and critical phenomena

$$S = \int_{\mathbb{R}^{4-\epsilon}} \frac{1}{2}(\partial\phi)^2 + \mu^{\epsilon}\frac{g_0}{4!}\phi^4 dx$$

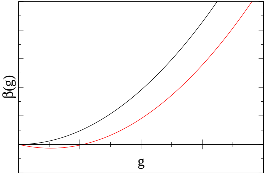
$$\beta(g) \equiv \mu\frac{dg}{d\mu} = -\epsilon g + \frac{3g^2}{(4\pi)^2}$$



# QFT and critical phenomena

$$S = \int_{\mathbb{R}^{4-\epsilon}} \frac{1}{2}(\partial\phi)^2 + \mu^\epsilon \frac{g_0}{4!}\phi^4 dx$$

$$\beta(g) \equiv \mu\frac{dg}{d\mu} = -\epsilon g + \frac{3g^2}{(4\pi)^2}$$



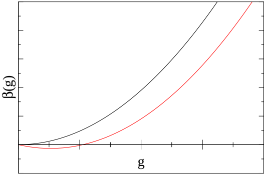

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{C_1}{|x_{12}|^{d-2}}$$

$$\langle\phi(x_1)\phi(x_2)\phi^2(x_3)\rangle = \frac{C_2}{|x_{13}|^{d-2}|x_{23}|^{d-2}}\left[1 + \frac{\epsilon}{3}\log\left(\frac{\mu^{-1}|x_{12}|}{|x_{13}||x_{23}|}\right)\right]$$

# QFT and critical phenomena

$$S = \int_{\mathbb{R}^{4-\epsilon}} \frac{1}{2}(\partial\phi)^2 + \mu^{\epsilon}\frac{g_0}{4!}\phi^4 dx$$

$$\beta(g) \equiv \mu\frac{dg}{d\mu} = -\epsilon g + \frac{3g^2}{(4\pi)^2}$$



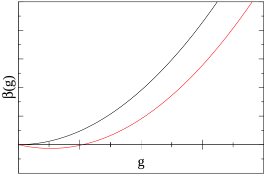

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{C_1}{|x_{12}|^{d-2}}$$

$$\left\langle\phi(x_1)\phi(x_2)\phi^2(x_3)\right\rangle = \frac{C_2\mu^{-\epsilon/3}}{|x_{12}|^{-\epsilon/3}|x_{13}|^{d-2+\epsilon/3}|x_{23}|^{d-2+\epsilon/3}}$$

# QFT and critical phenomena

$$S = \int_{\mathbb{R}^{4-\epsilon}} \frac{1}{2}(\partial\phi)^2 + \mu^{\epsilon}\frac{g_0}{4!}\phi^4 dx$$

$$\beta(g) \equiv \mu\frac{dg}{d\mu} = -\epsilon g + \frac{3g^2}{(4\pi)^2}$$




$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{C_1}{|x_{12}|^{d-2}}$$

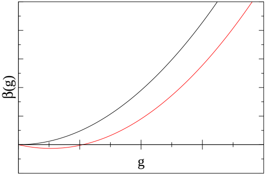
$$\left\langle\phi(x_1)\phi(x_2)\phi^2(x_3)\right\rangle = \frac{C_2\mu^{-\epsilon/3}}{|x_{12}|^{-\epsilon/3}|x_{13}|^{d-2+\epsilon/3}|x_{23}|^{d-2+\epsilon/3}}$$

$$\phi(x_1)\phi(x_2) = \lambda_{\phi\phi\phi^2}|x_{12}|^{\epsilon/3}\phi^2(x_2) + \dots \quad , \quad (no\ e^{-L/|x_{12}|})$$

# QFT and critical phenomena

$$S = \int_{\mathbb{R}^{4-\epsilon}} \frac{1}{2}(\partial\phi)^2 + \mu^{\epsilon}\frac{g_0}{4!}\phi^4 dx$$

$$\beta(g) \equiv \mu\frac{dg}{d\mu} = -\epsilon g + \frac{3g^2}{(4\pi)^2}$$




$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{C_1}{|x_{12}|^{d-2}} \quad , \quad \Delta_{\phi^n} = n\frac{d-2}{2} + \epsilon\frac{n(n-1)}{6}$$

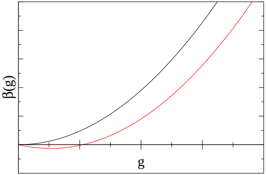
$$\langle\phi(x_1)\phi(x_2)\phi^2(x_3)\rangle = \frac{C_2\mu^{-\epsilon/3}}{|x_{12}|^{-\epsilon/3}|x_{13}|^{d-2+\epsilon/3}|x_{23}|^{d-2+\epsilon/3}}$$

$$\phi(x_1)\phi(x_2) = \lambda_{\phi\phi\phi^2}|x_{12}|^{\epsilon/3}\phi^2(x_2) + \dots \quad , \quad (no\ e^{-L/|x_{12}|})$$

# QFT and critical phenomena

$$S = \int_{\mathbb{R}^{4-\epsilon}} \frac{1}{2}(\partial\phi)^2 + \mu^{\epsilon}\frac{g_0}{4!}\phi^4 dx$$

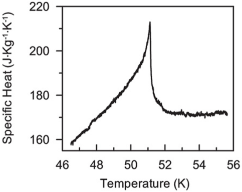
$$\beta(g) \equiv \mu\frac{dg}{d\mu} = -\epsilon g + \frac{3g^2}{(4\pi)^2}$$

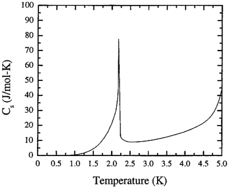



$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{C_1}{|x_{12}|^{d-2}} \quad , \quad \Delta_{\phi^n} = n\frac{d-2}{2} + \epsilon\frac{n(n-1)}{6}$$

$$\langle\phi(x_1)\phi(x_2)\phi^2(x_3)\rangle = \frac{C_2\mu^{-\epsilon/3}}{|x_{12}|^{-\epsilon/3}|x_{13}|^{d-2+\epsilon/3}|x_{23}|^{d-2+\epsilon/3}}$$

$$\phi(x_1)\phi(x_2) = \lambda_{\phi\phi\phi^2}|x_{12}|^{\epsilon/3}\phi^2(x_2) + \dots \quad , \quad (no\ e^{-L/|x_{12}|})$$




[Oleaga, Salazar, Bunkov; 2014]

[Donnelly, Barenghi; 1998]

# Conformal symmetry

Scale invariance enhances to conformal invariance which additionally includes special conformal generator $K_\mu = I \circ P_\mu \circ I$.

$$\langle 0 | \Phi(\infty) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \Phi(0) | 0 \rangle = \langle \Phi | \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) | \Phi \rangle$$

$$\phi_1(x)\phi_2(0) = \sum_{\mathcal{O}} \frac{\lambda_{12\mathcal{O}}}{|x|^{\Delta_1+\Delta_2-\Delta}} C(x,\partial)\mathcal{O}(0)$$

# Conformal symmetry

Scale invariance enhances to conformal invariance which additionally includes special conformal generator $K_\mu = I \circ P_\mu \circ I$.

$$\langle 0|\Phi(\infty)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\Phi(0)|0\rangle = \langle \Phi|\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)|\Phi\rangle$$

$$\phi_1(x)\phi_2(0) = \sum_{\mathcal{O}} \frac{\lambda_{12\mathcal{O}}}{|x|^{\Delta_1+\Delta_2-\Delta}} C(x,\partial)\mathcal{O}(0)$$

Correlators expressed with $I^{\mu\nu}_{ij} \equiv \delta^{\mu\nu} - 2\frac{x^\mu_{ij}x^\nu_{ij}}{|x_{ij}|^2}$, $Z^\mu_k \equiv \frac{x^\mu_{ik}}{|x_{ik}|^2} - \frac{x^\mu_{jk}}{|x_{jk}|^2}$.

$$\begin{aligned}
\langle \mathcal{O}^{\mu_1\dots\mu_\ell}(x_1)\mathcal{O}^{\nu_1\dots\nu_\ell}(x_2)\rangle &= \frac{I^{\mu_1\nu_1}_{12}\dots I^{\mu_\ell\nu_\ell}_{12} - \mathit{traces}}{|x_{12}|^{2\Delta}} \\
\langle \phi_1(x_1)\phi_2(x_2)\mathcal{O}^{\mu_1\dots\mu_\ell}(x_3)\rangle &= \frac{\lambda_{12\mathcal{O}} Z^{\mu_1}_3 \dots Z^{\mu_\ell}_3 - \mathit{traces}}{|x_{12}|^{\Delta_1+\Delta_2-\tau}|x_{13}|^{\tau+\Delta_{12}}|x_{23}|^{\tau-\Delta_{12}}}
\end{aligned}$$

# Conformal symmetry

Scale invariance enhances to conformal invariance which additionally includes special conformal generator $K_\mu = I \circ P_\mu \circ I$.

$$\langle 0|\Phi(\infty)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\Phi(0)|0\rangle = \langle\Phi|\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)|\Phi\rangle$$

$$\phi_1(x)\phi_2(0) = \sum_{\mathcal{O}} \frac{\lambda_{12\mathcal{O}}}{|x|^{\Delta_1+\Delta_2-\Delta}} C(x,\partial)\mathcal{O}(0)$$

Correlators expressed with $I^{\mu\nu}_{ij} \equiv \delta^{\mu\nu} - 2\frac{x^\mu_{ij}x^\nu_{ij}}{|x_{ij}|^2}$, $Z^\mu_k \equiv \frac{x^\mu_{ik}}{|x_{ik}|^2} - \frac{x^\mu_{jk}}{|x_{jk}|^2}$.

$$\begin{aligned}
\langle\mathcal{O}^{\mu_1\dots\mu_\ell}(x_1)\mathcal{O}^{\nu_1\dots\nu_\ell}(x_2)\rangle &= \frac{I^{\mu_1\nu_1}_{12}\dots I^{\mu_\ell\nu_\ell}_{12} - \textit{traces}}{|x_{12}|^{2\Delta}} \\
\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}^{\mu_1\dots\mu_\ell}(x_3)\rangle &= \frac{\lambda_{12\mathcal{O}}Z^{\mu_1}_3\dots Z^{\mu_\ell}_3 - \textit{traces}}{|x_{12}|^{\Delta_1+\Delta_2-\tau}|x_{13}|^{\tau+\Delta_{12}}|x_{23}|^{\tau-\Delta_{12}}} \\
\langle V^\mu(x_1)V^\nu(x_2)\phi(x_3)\rangle &= \frac{\lambda^{(1)}_{VV\phi}I^{\mu\nu}_{12} + \lambda^{(2)}_{VV\phi}|x_{12}|^2 Z^\mu_1 Z^\nu_2 - \textit{traces}}{|x_{12}|^{2\Delta_V-\Delta_\phi}|x_{13}|^{\Delta_\phi}|x_{23}|^{\Delta_\phi}}
\end{aligned}$$

# Conformal bootstrap

Four points can involve $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{G(u, v)}{|x_{12}|^{2\Delta_\phi} |x_{34}|^{2\Delta_\phi}}$$

# Conformal bootstrap

Four points can involve $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{G(u,v)}{|x_{12}|^{2\Delta_\phi} |x_{34}|^{2\Delta_\phi}} =$$

$$\sum_{\mathcal{O}} \sum_{n} \frac{\langle 0|\phi(x_1)\phi(x_2)|\partial^n \mathcal{O}\rangle \langle \partial^n \mathcal{O}|\phi(x_3)\phi(x_4)|0\rangle}{\langle \partial^n \mathcal{O}|\partial^n \mathcal{O}\rangle}$$

# Conformal bootstrap

Four points can involve $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{G(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}} =$$

$$\sum_{\mathcal{O}} \sum_{n} \frac{\langle 0|\phi(x_1)\phi(x_2)|\partial^n \mathcal{O}\rangle \langle \partial^n \mathcal{O}|\phi(x_3)\phi(x_4)|0\rangle}{\langle \partial^n \mathcal{O}|\partial^n \mathcal{O}\rangle}$$

Equating two decompositions yields crossing symmetry.

$$\sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} G_{\mathcal{O}}(u,v) = G(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} G_{\mathcal{O}}(v,u)$$

# Conformal bootstrap

Four points can involve $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{G(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}} =$$

$$\sum_{\mathcal{O}} \sum_n \frac{\langle 0|\phi(x_1)\phi(x_2)|\partial^n \mathcal{O}\rangle \langle \partial^n \mathcal{O}|\phi(x_3)\phi(x_4)|0\rangle}{\langle \partial^n \mathcal{O}|\partial^n \mathcal{O}\rangle}$$

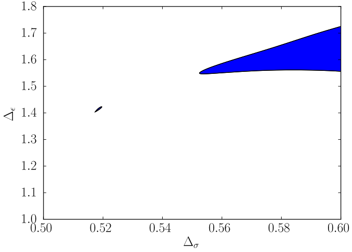
Equating two decompositions yields crossing symmetry.

$$\sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} G_{\mathcal{O}}(u,v) = G(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} G_{\mathcal{O}}(v,u)$$

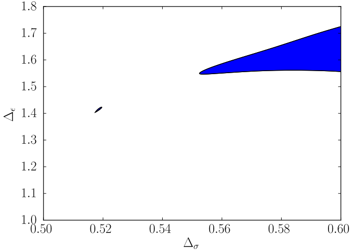
$$\sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} \left[ v^{\Delta_\phi} G_{\mathcal{O}}(u,v) - u^{\Delta_\phi} G_{\mathcal{O}}(v,u) \right] = 0$$

Look for functionals that are positive on all the blocks of a trial spectrum [Rattazzi, Rychkov, Tonni, Vichi; 0807.0004] .

# Famous results



# Famous results



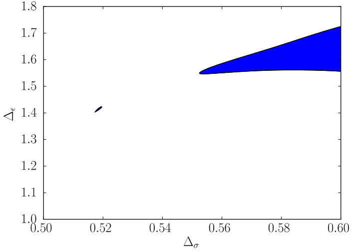

Island from $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$

[Kos, Poland, Simmons-Duffin, Vichi; 1603.04436] .

$$\begin{aligned} \Delta_\sigma &= 0.5181489(10) \\ \Delta_\epsilon &= 1.412625(10) \end{aligned}$$

Truncation kept 82 spins and functionals with 1265 components.

# Famous results




Island from $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$

[Kos, Poland, Simmons-Duffin, Vichi; 1603.04436] .

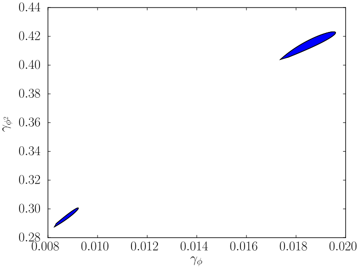
$$\begin{aligned}\Delta_\sigma &= 0.5181489(10)\\ \Delta_\epsilon &= 1.412625(10)\end{aligned}$$

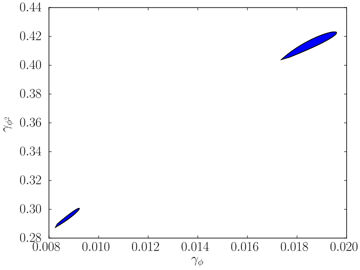
Truncation kept 82 spins and functionals with 1265 components.

---

- What is different?
- Bounds are "rigorous".
- Separation between "external" and "internal" information:
  - Hard to impose vs easy to impose.
  - Analytic vs non-analytic effect on the bounds.
  - Specify what a theory contains vs what it does not contain.

# Paradoxes with unitarity

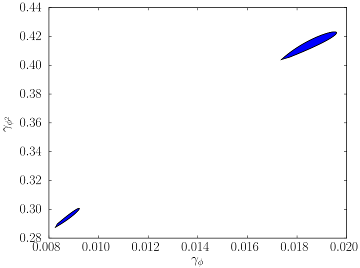





Build operator $\mathcal{O}_n$ by anti-symmetrizing $n > d$ indices.

$$\langle \mathcal{O}_n(\infty)\mathcal{O}_n(0)\rangle \propto \prod_{j=1}^{n-1}(d-j)$$

# Paradoxes with unitarity



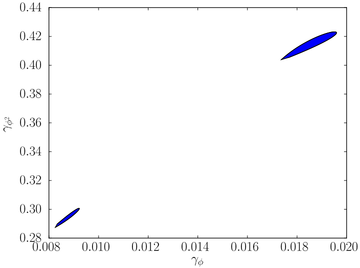

Build operator $\mathcal{O}_n$ by anti-symmetrizing $n > d$ indices.

$$\langle \mathcal{O}_n(\infty)\mathcal{O}_n(0)\rangle \propto \prod_{j=1}^{n-1}(d-j)$$

Happens for Wilson-Fisher in $d = 4 - \epsilon$ [Hogervorst, Rychkov, van Rees; 1512.00013] .

$$\mathcal{O}_5 = \delta^{[\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_3}\delta^{\mu_4}_{\nu_4}\delta^{\mu_5]}_{\nu_5]}\partial_{\mu_1}\partial^{\nu_1}\phi\partial_{\mu_2}\partial^{\nu_2}\phi\partial_{\mu_3}\partial^{\nu_3}\phi\partial_{\mu_4}\partial^{\nu_4}\phi\partial_{\mu_5}\partial^{\nu_5}\phi$$

# Paradoxes with unitarity




Build operator $\mathcal{O}_n$ by anti-symmetrizing $n > d$ indices.

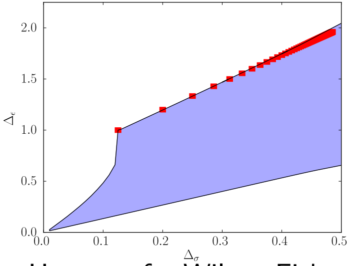
$$\langle \mathcal{O}_n(\infty)\mathcal{O}_n(0)\rangle \propto \prod_{j=1}^{n-1}(d-j)$$

Happens for Wilson-Fisher in $d = 4 - \epsilon$ [Hogervorst, Rychkov, van Rees; 1512.00013] .

$$\mathcal{O}_5 = \delta^{[\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_3}\delta^{\mu_4}_{\nu_4}\delta^{\mu_5]}_{\nu_5]}\partial_{\mu_1}\partial^{\nu_1}\phi\partial_{\mu_2}\partial^{\nu_2}\phi\partial_{\mu_3}\partial^{\nu_3}\phi\partial_{\mu_4}\partial^{\nu_4}\phi\partial_{\mu_5}\partial^{\nu_5}\phi$$

- Interesting to consider fermionic fixed-points with $\bar{\psi}\Gamma^{\mu_1\dots\mu_n}\psi$.
- Also 2D models with $c = 1 - 6\frac{(p-q)^2}{pq}$ [CB, 1712.06622] .
- $(p,q) = (m, m+1) \in \mathbb{N} \Rightarrow$ *Unitary MM*
- $(p,q) = (m, m+1) \in \mathbb{R} \Rightarrow$ *Generalized MM*

# Paradoxes with unitarity




Build operator $\mathcal{O}_n$ by anti-symmetrizing $n > d$ indices.

$$\langle \mathcal{O}_n(\infty)\mathcal{O}_n(0)\rangle \propto \prod_{j=1}^{n-1}(d-j)$$

Happens for Wilson-Fisher in $d = 4 - \epsilon$ [Hogervorst, Rychkov, van Rees; 1512.00013] .

$$\mathcal{O}_5 = \delta^{[\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_3}\delta^{\mu_4}_{\nu_4}\delta^{\mu_5]}_{\nu_5]}\partial_{\mu_1}\partial^{\nu_1}\phi\partial_{\mu_2}\partial^{\nu_2}\phi\partial_{\mu_3}\partial^{\nu_3}\phi\partial_{\mu_4}\partial^{\nu_4}\phi\partial_{\mu_5}\partial^{\nu_5}\phi$$

- Interesting to consider fermionic fixed-points with $\bar{\psi}\Gamma^{\mu_1\cdots\mu_n}\psi$.
- Also 2D models with $c = 1 - 6\frac{(p-q)^2}{pq}$ [CB, 1712.06622] .
- $(p, q) = (m, m+1) \in \mathbb{N} \Rightarrow$ *Unitary MM*
- $(p, q) = (m, m+1) \in \mathbb{R} \Rightarrow$ *Generalized MM*

# Evanescent operators with fermions

Three renormalizable QFTs in $d = 2$ (expect enhanced symmetry).

$$S = \int \bar{\psi}_i \partial\!\!\!/ \psi^i - \frac{1}{2} g_S (\bar{\psi}\psi)^2 - \frac{1}{2} g_V (\bar{\psi}\gamma^\mu \psi)^2 - \frac{1}{2} g_P (\bar{\psi}\gamma_5 \psi)^2 dx$$

# Evanescent operators with fermions

Three renormalizable QFTs in $d = 2$ (expect enhanced symmetry).

$$
\begin{aligned}
S &= \int \bar{\psi}_i \partial\!\!\!/ \psi^i - \frac{1}{2} g_S (\bar{\psi}\psi)^2 - \frac{1}{2} g_V (\bar{\psi}\gamma^\mu\psi)^2 - \frac{1}{2} g_P (\bar{\psi}\gamma_5\psi)^2 dx \\
S &= \int \bar{\psi}_i \partial\!\!\!/ \psi^i + \sum_{m=0}^{\infty} g_m \delta^i_j \delta^k_l \left( \bar{\psi}_i \Gamma^{(m)} \psi^j \right) \left( \bar{\psi}_k \Gamma^{(m)} \psi^l \right) dx
\end{aligned}
$$

Potentially many CFTs in $d = 2 + \epsilon$ (expect $U(N)$ symmetry).

# Evanescent operators with fermions

Three renormalizable QFTs in $d = 2$ (expect enhanced symmetry).

$$S = \int \bar{\psi}_i \partial\!\!\!/ \psi^i - \frac{1}{2} g_S (\bar{\psi}\psi)^2 - \frac{1}{2} g_V (\bar{\psi}\gamma^\mu\psi)^2 - \frac{1}{2} g_P (\bar{\psi}\gamma_5\psi)^2 dx$$

$$S = \int \bar{\psi}_i \partial\!\!\!/ \psi^i + \sum_{m=0}^{\infty} T(m)^{ik}{}_{jl} \left(\bar{\psi}_i \Gamma^{(m)} \psi^j\right) \left(\bar{\psi}_k \Gamma^{(m)} \psi^l\right) dx$$

Potentially many CFTs in $d = 2 + \epsilon$ (nice for a single $T(m)$).

# Evanescent operators with fermions

Three renormalizable QFTs in $d = 2$ (expect enhanced symmetry).

$$\begin{aligned} S &= \int \bar{\psi}_i \partial\!\!\!/ \psi^i - \frac{1}{2} g_S (\bar{\psi}\psi)^2 - \frac{1}{2} g_V (\bar{\psi}\gamma^\mu\psi)^2 - \frac{1}{2} g_P (\bar{\psi}\gamma_5\psi)^2 dx \\ S &= \int \bar{\psi}_i \partial\!\!\!/ \psi^i + \sum_{m=0}^{\infty} T(m)^{ik}{}_{jl} \left(\bar{\psi}_i \Gamma^{(m)} \psi^j\right) \left(\bar{\psi}_k \Gamma^{(m)} \psi^l\right) dx \end{aligned}$$

Potentially many CFTs in $d = 2 + \epsilon$ (nice for a single $T(m)$).

$$S = \int \bar{\psi} \partial\!\!\!/ \psi - \frac{1}{2} g \left[ (\bar{\psi}_i \psi^j)(\bar{\psi}_j \psi^i) - (\bar{\psi}_i \psi^j)(\bar{\psi}_i \psi^j) \right] dx$$

# Evanescent operators with fermions

Three renormalizable QFTs in $d = 2$ (expect enhanced symmetry).

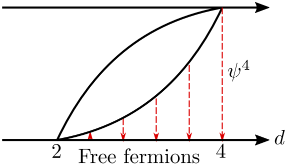
$$S = \int \bar{\psi}_i \partial\!\!\!/ \psi^i - \frac{1}{2} g_S (\bar{\psi}\psi)^2 - \frac{1}{2} g_V (\bar{\psi}\gamma^\mu\psi)^2 - \frac{1}{2} g_P (\bar{\psi}\gamma_5\psi)^2 dx$$

$$S = \int \bar{\psi}_i \partial\!\!\!/ \psi^i + \sum_{m=0}^{\infty} T(m)^{ik}{}_{jl} \left( \bar{\psi}_i \Gamma^{(m)} \psi^j \right) \left( \bar{\psi}_k \Gamma^{(m)} \psi^l \right) dx$$

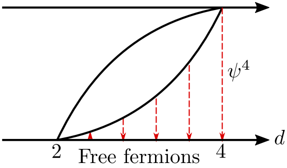
Potentially many CFTs in $d = 2 + \epsilon$ (nice for a single $T(m)$).

$$\begin{aligned} S &= \int \bar{\psi} \partial\!\!\!/ \psi - \frac{1}{2} g \left[ (\bar{\psi}_i \psi^j)(\bar{\psi}_j \psi^i) - (\bar{\psi}_i \psi^j)(\bar{\psi}_i \psi^j) \right] dx \\ &\neq \int \bar{\psi} \partial\!\!\!/ \psi - \frac{1}{2} g \left[ \frac{1}{2} (\bar{\psi}_i \psi^i)(\bar{\psi}_j \psi^j) + \frac{1}{2} (\bar{\psi}_i \gamma^\mu \psi^i)(\bar{\psi}_j \gamma_\mu \psi^j) \right] \\ &\quad + \frac{1}{2} g \left[ \frac{1}{4} (\bar{\psi}_i \gamma^{\mu\nu} \psi^i)(\bar{\psi}_j \gamma_{\mu\nu} \psi^j) + (\bar{\psi}_i \psi^j)(\bar{\psi}_i \psi^j) \right] dx \end{aligned}$$

# Dualities



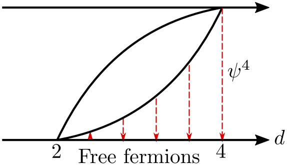
# Dualities




Introduce Lagrange-multiplier $\sigma$ in GN model:

$$S = \int \bar{\psi}\not{\partial}\psi + g\sigma\bar{\psi}\psi + \frac{1}{2}g\sigma^2 dx$$

# Dualities



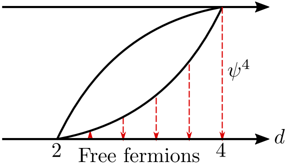

Introduce Lagrange-multiplier $\sigma$ in GN model:

$$S = \int \bar{\psi}\partial\!\!\!/\psi + g\sigma\bar{\psi}\psi + \frac{1}{2}g\sigma^2 dx$$

$$S = \int \bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}(\partial\sigma)^2 + g\sigma\bar{\psi}\psi + \frac{\lambda}{4!}\sigma^4 dx$$

Situation may hold for $SO(2) \times U(1)$ as well.

# Dualities




Introduce Lagrange-multiplier $\sigma$ in GN model:

$$S = \int \bar{\psi}\partial\!\!\!/\psi + g\sigma\bar{\psi}\psi + \frac{1}{2}g\sigma^2 dx$$

$$S = \int \bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}(\partial\sigma)^2 + g\sigma\bar{\psi}\psi + \frac{\lambda}{4!}\sigma^4 dx$$

Situation may hold for $SO(2) \times U(1)$ as well.

---

- From pure thought: T-duality, 1D and 2D bosonization.
- QCD in $d < 4$ from $U(N_f) \times SU(N_c)$ Thirring [Hasenfratz$^2$; 9207017] .
- Seiberg duality is $N_c \leftrightarrow N_f - N_c$ in $\mathcal{N} = 1$ Super-QCD [Seiberg; 9411149] .
- Argyres-Douglas CFTs in $\mathcal{N} = 2$ [Argyres, Plesser, Seiberg, Witten; 9511154] .

# What is locality?

Cheap answer: the existence of local operators.

# What is locality?

Cheap answer: the existence of local operators.

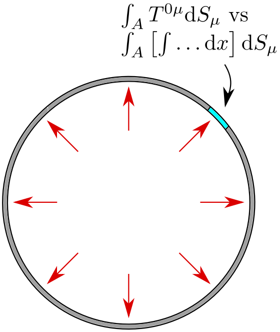
$\begin{cases} E = \int_{\mathbb{R}^{d-1}} T^{00} dx \\ \partial_\mu T^{0\mu} = 0 \end{cases}$ implies more than just $\partial_0 E = 0$. Computing $\partial_0 E\big|_B$ should only require knowledge about $\partial B$.

# What is locality?

Cheap answer: the existence of local operators.

$$\begin{cases} E = \int_{\mathbb{R}^{d-1}} T^{00} dx \\ \partial_\mu T^{0\mu} = 0 \end{cases}$$

implies more than just $\partial_0 E = 0$. Computing $\partial_0 E\big|_B$ should only require knowledge about $\partial B$.



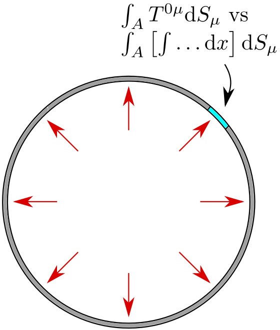
# What is locality?

Cheap answer: the existence of local operators.

$$\begin{cases} E = \int_{\mathbb{R}^{d-1}} T^{00} dx \\ \partial_\mu T^{0\mu} = 0 \end{cases}$$

implies more than just $\partial_0 E = 0$. Computing $\partial_0 E\big|_B$ should only require knowledge about $\partial B$.

1. $\exists$ a stress tensor.
2. $\exists$ a conserved current for every symmetry.
3. Generalized modular invariance.



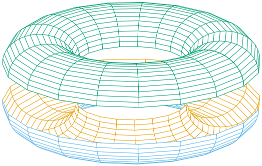
# What is locality?

Cheap answer: the existence of local operators.

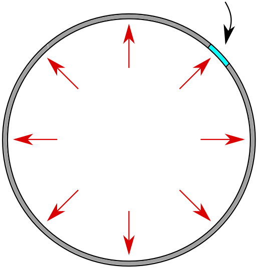
$$\begin{cases} E = \int_{\mathbb{R}^{d-1}} T^{00} dx \\ \partial_\mu T^{0\mu} = 0 \end{cases}$$

implies more than just $\partial_0 E = 0$. Computing $\partial_0 E\big|_B$ should only require knowledge about $\partial B$.

1. $\exists$ a stress tensor.
2. $\exists$ a conserved current for every symmetry.
3. Generalized modular invariance.



$\int_A T^{0\mu} \mathrm{d}S_\mu$ vs
$\int_A \left[\int \ldots \mathrm{d}x\right] \mathrm{d}S_\mu$



Consistency of 2D CFT on a torus guarantees consistency for any Riemann surface [Moore, Seiberg; 89] .

# Deforming a generalized free theory

Consider free field in $\mathbb{R}^4$ with $\langle \phi(x) \phi(0) \rangle = \frac{1}{|x|^2}$.

# Deforming a generalized free theory

Consider free field in $\mathbb{R}^4$ with $\langle \phi(x)\phi(0) \rangle = \frac{1}{|x|^2}$.
Restricting to $(x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, 0)$,

| $d = 4$ point of view | $d = 3$ point of view |
|---|---|
| $\partial_\mu T^{\mu\nu} = 0$ | $\partial_\mu T^{\mu\nu} = -\partial_\perp T^{\perp\nu}$ |
| $\Delta_T = d$ | $\Delta_T = d + 1$ |
| $\Delta_\phi = \frac{d-2}{2}$ | $\Delta_\phi = \frac{d-1}{2}$ |
| $\langle \phi(x)\phi(0) \rangle = \frac{1}{\lvert x\rvert^{d-2}}$ | $\langle \phi(y)\phi(0) \rangle = \frac{1}{\lvert y\rvert^{d-1}}$ |

# Deforming a generalized free theory

Consider free field in $\mathbb{R}^4$ with $\langle \phi(x)\phi(0) \rangle = \frac{1}{|x|^2}$.
Restricting to $(x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, 0)$,

| $d = 4$ point of view | $d = 3$ point of view |
|---|---|
| $\partial_\mu T^{\mu\nu} = 0$ | $\partial_\mu T^{\mu\nu} = -\partial_\perp T^{\perp\nu}$ |
| $\Delta_T = d$ | $\Delta_T = d + 1$ |
| $\Delta_\phi = \frac{d-2}{2}$ | $\Delta_\phi = \frac{d-1}{2}$ |
| $\langle \phi(x)\phi(0) \rangle = \frac{1}{\lvert x\rvert^{d-2}}$ | $\langle \phi(y)\phi(0) \rangle = \frac{1}{\lvert y\rvert^{d-1}}$ |

Local + free $\Rightarrow$ nonlocal + free, when restricted to a slice.

$$\begin{aligned} S &= \int_{\mathbb{R}^{d+2-s}} -\frac{1}{2}\phi\partial^2\phi dx \\ S_\partial &= \int_{\mathbb{R}^d} -\frac{1}{2}\phi\partial^s\phi dy \end{aligned}$$

# Deforming a generalized free theory

Consider free field in $\mathbb{R}^4$ with $\langle \phi(x)\phi(0) \rangle = \frac{1}{|x|^2}$.
Restricting to $(x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, 0)$,

| $d = 4$ point of view | $d = 3$ point of view |
|---|---|
| $\partial_\mu T^{\mu\nu} = 0$ | $\partial_\mu T^{\mu\nu} = -\partial_\perp T^{\perp\nu}$ |
| $\Delta_T = d$ | $\Delta_T = d + 1$ |
| $\Delta_\phi = \frac{d-2}{2}$ | $\Delta_\phi = \frac{d-1}{2}$ |
| $\langle \phi(x)\phi(0) \rangle = \frac{1}{\lvert x\rvert^{d-2}}$ | $\langle \phi(y)\phi(0) \rangle = \frac{1}{\lvert y\rvert^{d-1}}$ |

Local + free $\Rightarrow$ nonlocal + free, when restricted to a slice.

$$\begin{aligned} S &= \int_{\mathbb{R}^{d+2-s}} -\frac{1}{2}\phi\partial^2\phi dx + \int_{\mathbb{R}^d} \frac{\lambda}{4!}\phi^4 dy \\ S_\partial &= \int_{\mathbb{R}^d} -\frac{1}{2}\phi\partial^s\phi + \frac{\lambda}{4!}\phi^4 dy \end{aligned}$$

# Deforming a generalized free theory

Consider free field in $\mathbb{R}^4$ with $\langle \phi(x)\phi(0) \rangle = \frac{1}{|x|^2}$.
Restricting to $(x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, 0)$,

| $d = 4$ point of view | $d = 3$ point of view |
|---|---|
| $\partial_\mu T^{\mu\nu} = 0$ | $\partial_\mu T^{\mu\nu} = -\partial_\perp T^{\perp\nu}$ |
| $\Delta_T = d$ | $\Delta_T = d + 1$ |
| $\Delta_\phi = \frac{d-2}{2}$ | $\Delta_\phi = \frac{d-1}{2}$ |
| $\langle \phi(x)\phi(0) \rangle = \frac{1}{\lvert x\rvert^{d-2}}$ | $\langle \phi(y)\phi(0) \rangle = \frac{1}{\lvert y\rvert^{d-1}}$ |

Local + free $\Rightarrow$ nonlocal + free, when restricted to a slice.

$$\begin{aligned} S &= \int_{\mathbb{R}^{d+2-s}} -\frac{1}{2}\phi\partial^2\phi dx + \int_{\mathbb{R}^d} \frac{\lambda}{4!}\phi^4 dy \\ S_\partial &= \int_{\mathbb{R}^d} -\frac{1}{2}\phi\partial^s\phi + \frac{\lambda}{4!}\phi^4 dy \end{aligned}$$

All signs of conformal invariance in IR [Paulos, Rychkov, van Rees, Zan; 1509.00008] .

# The long-range Ising model

$$H_{LRI} = -J \sum_{i \neq j} \frac{\sigma_i \sigma_j}{|i - j|^{d+s}}$$

$$S = -\iint \frac{\phi(x)\phi(y)}{|x - y|^{d+s}} dydx + \int \frac{\lambda}{4!} \phi^4(x) dx$$

# The long-range Ising model

$$H_{LRI} = -J \sum_{i \neq j} \frac{\sigma_i \sigma_j}{|i-j|^{d+s}}$$

$$S = -\iint \frac{\phi(x)\phi(y)}{|x-y|^{d+s}} dy dx + \int \frac{\lambda}{4!} \phi^4(x) dx$$

- Second-order phase transition if $1 \leq d < 4$ [Dyson; 69] .
- $\epsilon = 2s - d$ expansion works for $\frac{d}{2} < s < s_*$ [Fisher, Ma, Nickel; 72] .
- MC estimates in 1D, 2D [Angelini, Parisi, Ricci-Tersenghi; 1401.6805] .

# The long-range Ising model

$$H_{LRI} = -J \sum_{i \neq j} \frac{\sigma_i \sigma_j}{|i-j|^{d+s}}$$

$$S = -\iint \frac{\phi(x)\phi(y)}{|x-y|^{d+s}} dydx + \int \frac{\lambda}{4!} \phi^4(x) dx$$

- Second-order phase transition if $1 \leq d < 4$ [Dyson; 69] .
- $\epsilon = 2s - d$ expansion works for $\frac{d}{2} < s < s_*$ [Fisher, Ma, Nickel; 72] .
- MC estimates in 1D, 2D [Angelini, Parisi, Ricci-Tersenghi; 1401.6805] .

If $\iint \frac{\sigma(x)\sigma(y)}{|x-y|^{d+s}} dydx$ controls the approach to SRI at $s_*$, we should define $\chi(x) = \int \frac{\sigma(y)}{|x-y|^{d+s}} dy$.

# The long-range Ising model

$$H_{LRI} = -J \sum_{i \neq j} \frac{\sigma_i \sigma_j}{|i-j|^{d+s}}$$

$$S = -\iint \frac{\phi(x)\phi(y)}{|x-y|^{d+s}} dydx + \int \frac{\lambda}{4!}\phi^4(x)dx$$

- Second-order phase transition if $1 \leq d < 4$ [Dyson; 69] .
- $\epsilon = 2s - d$ expansion works for $\frac{d}{2} < s < s_*$ [Fisher, Ma, Nickel; 72] .
- MC estimates in 1D, 2D [Angelini, Parisi, Ricci-Tersenghi; 1401.6805] .

If $\iint \frac{\sigma(x)\sigma(y)}{|x-y|^{d+s}} dydx$ controls the approach to SRI at $s_*$, we should define $\chi(x) = \int \frac{\sigma(y)}{|x-y|^{d+s}} dy$.

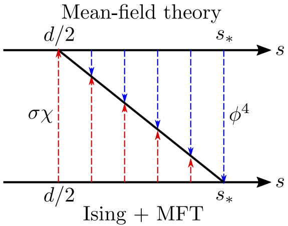
$$S = S_{SRI} - \iint \frac{\chi(x)\chi(y)}{|x-y|^{d-s}} dydx + \int g\sigma(x)\chi(x)dx$$

Enables $\delta = \frac{s_* - s}{2}$ perturbation as well [CB, Rastelli, Rychkov, Zan; 1703.05325] .

# The long-range Ising model

$$H_{LRI} = -J \sum_{i \neq j} \frac{\sigma_i \sigma_j}{|i-j|^{d+s}}$$

$$S = -\iint \frac{\phi(x)\phi(y)}{|x-y|^{d+s}} dydx + \int \frac{\lambda}{4!}\phi^4(x)dx$$




- Second-order phase transition if $1 \leq d < 4$ [Dyson; 69] .
- $\epsilon = 2s - d$ expansion works for $\frac{d}{2} < s < s_*$ [Fisher, Ma, Nickel; 72] .
- MC estimates in 1D, 2D [Angelini, Parisi, Ricci-Tersenghi; 1401.6805] .

If $\iint \frac{\sigma(x)\sigma(y)}{|x-y|^{d+s}} dydx$ controls the approach to SRI at $s_*$, we should define $\chi(x) = \int \frac{\sigma(y)}{|x-y|^{d+s}} dy$.

$$S = S_{SRI} - \iint \frac{\chi(x)\chi(y)}{|x-y|^{d-s}} dydx + \int g\sigma(x)\chi(x)dx$$

Enables $\delta = \frac{s_* - s}{2}$ perturbation as well [CB, Rastelli, Rychkov, Zan; 1703.05325] .

# Protected operators

$$\begin{array}{ccccc} \phi & \leftrightarrow & \sigma & & \phi^4 \leftrightarrow \sigma\chi \\ \phi^2 & \leftrightarrow & \epsilon & & [\phi\phi]^{\mu\nu}_{0,2} \leftrightarrow T^{\mu\nu} \\ \phi^3 & \leftrightarrow & \chi & & \partial_\nu[\phi\phi]^{\mu\nu}_{0,2} \leftrightarrow [\sigma\chi]^{\mu}_{0,1} \end{array}$$

# Protected operators

$$
\begin{array}{ccccc}
\phi & \leftrightarrow & \sigma & & \phi^4 \leftrightarrow \sigma\chi \\
\phi^2 & \leftrightarrow & \epsilon & & [\phi\phi]^{\mu\nu}_{0,2} \leftrightarrow T^{\mu\nu} \\
\phi^3 & \leftrightarrow & \chi & & \partial_\nu[\phi\phi]^{\mu\nu}_{0,2} \leftrightarrow [\sigma\chi]^{\mu}_{0,1}
\end{array}
$$

---

- Expect $\phi$ an $\chi$ to be fixed at their UV dimensions as they have nonlocal kinetic terms.
- In the usual Wilson-Fished fixed-point, $\partial^2\phi = \frac{\lambda}{3!}\phi^3$ and $\phi^3$ is a descendant since $\partial^2$ is a conformal generator.
- Instead, our EOMs are nonlocal so $\phi^3$ and $\chi$ are (heavily constrained) primaries.

# Protected operators

$$\begin{array}{ccccc} \phi & \leftrightarrow & \sigma & & \phi^4 \leftrightarrow \sigma\chi \\ \phi^2 & \leftrightarrow & \epsilon & & [\phi\phi]^{\mu\nu}_{0,2} \leftrightarrow T^{\mu\nu} \\ \phi^3 & \leftrightarrow & \chi & & \partial_\nu[\phi\phi]^{\mu\nu}_{0,2} \leftrightarrow [\sigma\chi]^{\mu}_{0,1} \end{array}$$

---

- Expect $\phi$ an $\chi$ to be fixed at their UV dimensions as they have nonlocal kinetic terms.
- In the usual Wilson-Fished fixed-point, $\partial^2\phi = \frac{\lambda}{3!}\phi^3$ and $\phi^3$ is a descendant since $\partial^2$ is a conformal generator.
- Instead, our EOMs are nonlocal so $\phi^3$ and $\chi$ are (heavily constrained) primaries.

$$\Delta_\phi = \frac{d-s}{2} \quad , \quad \Delta_\sigma = \frac{d-s}{2}$$

$$\Delta_{\phi^3} = \frac{d+s}{2} \quad , \quad \Delta_\chi = \frac{d+s}{2}$$

$$\partial^s\phi \sim \phi^3 \quad , \quad \partial^{-s}\chi \sim \sigma$$

# Unprotected operators

Two-loop dimensions of double-trace operators: [CB; 1810.07199]

$$\Delta_{[\phi\phi]_{0,\ell}} = \frac{d+4-\varepsilon}{2} - \frac{2\Gamma(\ell)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+\ell\right)}\left(\frac{\varepsilon}{3}\right)^2 + O(\varepsilon^3)$$

Harder one was already known: [Fisher, Ma, Nickel; 72]

$$\Delta_{\phi^2} = \frac{d-\varepsilon}{2} + \frac{\varepsilon}{3} + \left[\psi(1) - 2\psi\left(\frac{d}{4}\right) + \psi\left(\frac{d}{2}\right)\right]\left(\frac{\varepsilon}{3}\right)^2 + O(\varepsilon^3)$$

# Unprotected operators

Two-loop dimensions of double-trace operators: [CB; 1810.07199]

$$\Delta_{[\phi\phi]_{0,\ell}} = \frac{d+4-\varepsilon}{2} - \frac{2\Gamma(\ell)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+\ell\right)}\left(\frac{\varepsilon}{3}\right)^2 + O(\varepsilon^3)$$

Harder one was already known: [Fisher, Ma, Nickel; 72]

$$\Delta_{\phi^2} = \frac{d-\varepsilon}{2} + \frac{\varepsilon}{3} + \left[\psi(1) - 2\psi\left(\frac{d}{4}\right) + \psi\left(\frac{d}{2}\right)\right]\left(\frac{\varepsilon}{3}\right)^2 + O(\varepsilon^3)$$

Dual expressions in 2D using exact solution:

$$\begin{aligned} \Delta_T &= 2 + 3.65\delta + O(\delta^2) \\ \Delta_\epsilon &= 1 + O(\delta^2) \end{aligned}$$

Dual expressions in 3D using bootstrap data: [CB, Rastelli, Rychkov, Zan; 1703.05325]

$$\begin{aligned} \Delta_T &= 3 + 2.33\delta + O(\delta^2) \\ \Delta_\epsilon &= \Delta_\epsilon^{SRI} + 0.27\delta + O(\delta^2) \end{aligned}$$

# Protected OPE coefficients

Compute $\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\chi(0)\rangle$ by hitting $\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\sigma(0)\rangle$ with:

$$n_\chi(s)\chi(x) = \int \frac{n_\sigma(s)\sigma(y)}{|x-y|^{d+s}} dy$$

Result will look like $\frac{n_\sigma(s)}{n_\chi(s)} R_{12}\lambda_{12\sigma}$ but it must be equal to $\lambda_{12\chi}$!

# Protected OPE coefficients

Compute $\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\chi(0)\rangle$ by hitting $\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\sigma(0)\rangle$ with:

$$n_\chi(s)\chi(x) = \int \frac{n_\sigma(s)\sigma(y)}{|x-y|^{d+s}} dy$$

Result will look like $\frac{n_\sigma(s)}{n_\chi(s)} R_{12}\lambda_{12\sigma}$ but it must be equal to $\lambda_{12\chi}$!

$$\lambda^{(m)}_{12\chi}\lambda^{(n)}_{34\sigma} = \frac{\Gamma\left(\frac{\Delta_\chi \pm \Delta_{12} + \ell_1 + \ell_2 - 2m}{2}\right)\Gamma\left(\frac{\Delta_\sigma \pm \Delta_{34} + \ell_1 + \ell_2 - 2n}{2}\right)}{\Gamma\left(\frac{\Delta_\sigma \pm \Delta_{12} + \ell_1 + \ell_2 - 2m}{2}\right)\Gamma\left(\frac{\Delta_\chi \pm \Delta_{34} + \ell_1 + \ell_2 - 2n}{2}\right)}\lambda^{(m)}_{12\sigma}\lambda^{(n)}_{34\chi}$$

# Protected OPE coefficients

Compute $\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\chi(0)\rangle$ by hitting $\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\sigma(0)\rangle$ with:

$$n_\chi(s)\chi(x) = \int \frac{n_\sigma(s)\sigma(y)}{|x-y|^{d+s}} dy$$

Result will look like $\frac{n_\sigma(s)}{n_\chi(s)} R_{12}\lambda_{12\sigma}$ but it must be equal to $\lambda_{12\chi}$!

$$\lambda^{(m)}_{12\chi}\lambda^{(n)}_{34\sigma} = \frac{\Gamma\left(\frac{\Delta_\chi \pm \Delta_{12} + \ell_1 + \ell_2 - 2m}{2}\right)\Gamma\left(\frac{\Delta_\sigma \pm \Delta_{34} + \ell_1 + \ell_2 - 2n}{2}\right)}{\Gamma\left(\frac{\Delta_\sigma \pm \Delta_{12} + \ell_1 + \ell_2 - 2m}{2}\right)\Gamma\left(\frac{\Delta_\chi \pm \Delta_{34} + \ell_1 + \ell_2 - 2n}{2}\right)}\lambda^{(m)}_{12\sigma}\lambda^{(n)}_{34\chi}$$

$$\lambda^2_{\sigma\chi\mathcal{O}} = \frac{\Gamma\left(\frac{d-\Delta+\ell}{2}\right)^2\Gamma\left(\frac{d-2\Delta_\sigma+\Delta+\ell}{2}\right)\Gamma\left(\frac{2\Delta_\sigma-d+\Delta+\ell}{2}\right)}{\Gamma\left(\frac{\Delta+\ell}{2}\right)^2\Gamma\left(\frac{2\Delta_\sigma-\Delta+\ell}{2}\right)\Gamma\left(\frac{2d-2\Delta_\sigma-\Delta+\ell}{2}\right)}\lambda_{\sigma\sigma\mathcal{O}}\lambda_{\chi\chi\mathcal{O}}$$

# Protected OPE coefficients

Compute $\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\chi(0)\rangle$ by hitting $\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\sigma(0)\rangle$ with:

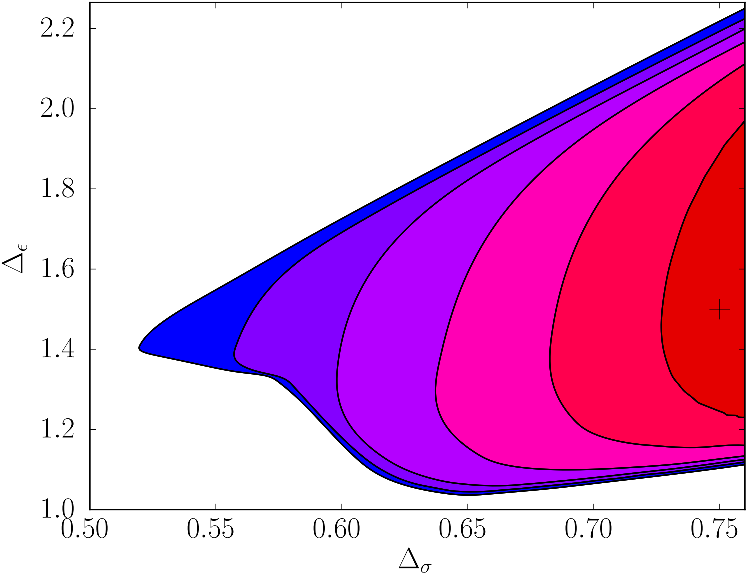
$$n_\chi(s)\chi(x) = \int \frac{n_\sigma(s)\sigma(y)}{|x-y|^{d+s}}dy$$

Result will look like $\frac{n_\sigma(s)}{n_\chi(s)}R_{12}\lambda_{12\sigma}$ but it must be equal to $\lambda_{12\chi}$!

$$\lambda^{(m)}_{12\chi}\lambda^{(n)}_{34\sigma} = \frac{\Gamma\left(\frac{\Delta_\chi \pm \Delta_{12}+\ell_1+\ell_2-2m}{2}\right)\Gamma\left(\frac{\Delta_\sigma \pm \Delta_{34}+\ell_1+\ell_2-2n}{2}\right)}{\Gamma\left(\frac{\Delta_\sigma \pm \Delta_{12}+\ell_1+\ell_2-2m}{2}\right)\Gamma\left(\frac{\Delta_\chi \pm \Delta_{34}+\ell_1+\ell_2-2n}{2}\right)}\lambda^{(m)}_{12\sigma}\lambda^{(n)}_{34\chi}$$

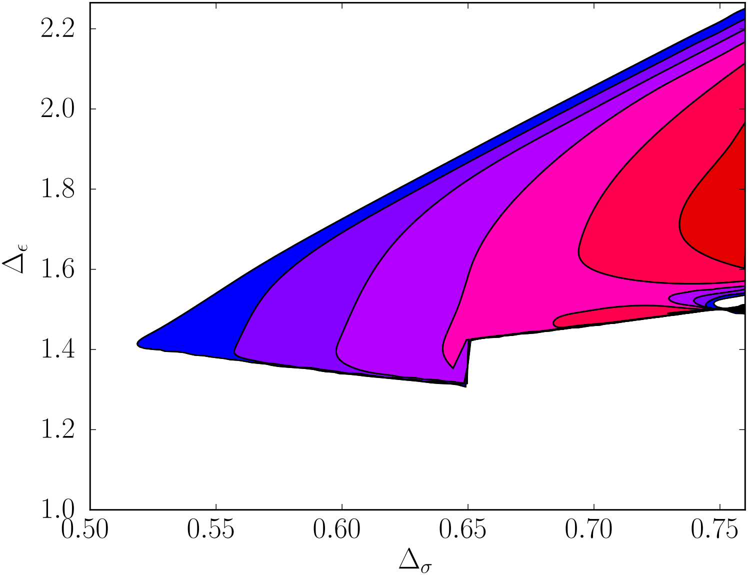
$$\lambda^2_{\sigma\chi\mathcal{O}} = \frac{\Gamma\left(\frac{d-\Delta+\ell}{2}\right)^2\Gamma\left(\frac{d-2\Delta_\sigma+\Delta+\ell}{2}\right)\Gamma\left(\frac{2\Delta_\sigma-d+\Delta+\ell}{2}\right)}{\Gamma\left(\frac{\Delta+\ell}{2}\right)^2\Gamma\left(\frac{2\Delta_\sigma-\Delta+\ell}{2}\right)\Gamma\left(\frac{2d-2\Delta_\sigma-\Delta+\ell}{2}\right)}\lambda_{\sigma\sigma\mathcal{O}}\lambda_{\chi\chi\mathcal{O}}$$

- Odd spin: $[\sigma\chi]_{n,\ell}$ cannot leave the pole by Bose symmetry.
- Even spin: $\lambda_{\sigma\sigma\mathcal{O}}\lambda_{\chi\chi\mathcal{O}}G^{0,0}_{\mathcal{O}}(u,v)$ and $\lambda^2_{\sigma\chi\mathcal{O}}G^{\Delta_{\chi\sigma},\Delta_{\sigma\chi}}_{\mathcal{O}}(u,v)$ combine into a superblock.



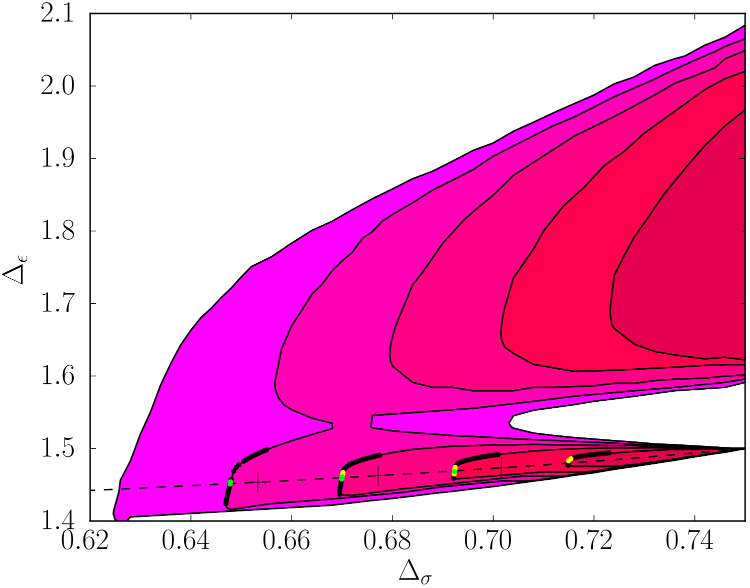

Mysterious regions with $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$.

# Numerical results




More interesting after adding $\langle \chi\chi\chi\chi \rangle$, $\langle \sigma\sigma\chi\chi \rangle$, $\langle \chi\chi\epsilon\epsilon \rangle$.

# Numerical results




Higher precision with spin-2 gaps of 3.25, 3.3, 3.35, 3.4, 3.45.

# Perturbation theory for a general beta function

$$\begin{aligned} \text{“}S\text{“} &\mapsto \text{“}S\text{“} + \int g\hat{\mathcal{O}}dx \\ \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle &\mapsto \left\langle \mathcal{O}_1 \dots \mathcal{O}_n e^{\int g\hat{\mathcal{O}}dx} \right\rangle \end{aligned}$$

Perturbations to correlators of $\hat{\mathcal{O}}$ itself compute $\beta(g)$.

# Perturbation theory for a general beta function

$$\text{“}S\text{“} \mapsto \text{“}S\text{“} + \int g\hat{\mathcal{O}} dx$$

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \mapsto \left\langle \mathcal{O}_1 \dots \mathcal{O}_n e^{\int g\hat{\mathcal{O}} dx} \right\rangle$$

Perturbations to correlators of $\hat{\mathcal{O}}$ itself compute $\beta(g)$.

For exactly marginal operator [Bashmakov, Bertolini, Raj; 1709.01749] [CB; 1709.03967] ,

$$\beta_0 = 0 \Rightarrow \hat{\Delta} = d$$

$$\beta_1 = 0 \Rightarrow \lambda_{\hat{\mathcal{O}}\hat{\mathcal{O}}\hat{\mathcal{O}}} = 0$$

$$\beta_2 = 0 \Rightarrow \sum_{\mathcal{O}} \lambda^2_{\hat{\mathcal{O}}\hat{\mathcal{O}}\mathcal{O}} \int_S G_{\mathcal{O}}(u, v)|x|^{-2d} dx = 0$$

# Perturbation theory for a general beta function

$$\text{“}S\text{“} \mapsto \text{“}S\text{“} + \int g\hat{\mathcal{O}}dx$$

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \mapsto \left\langle \mathcal{O}_1 \dots \mathcal{O}_n e^{\int g\hat{\mathcal{O}}dx} \right\rangle$$

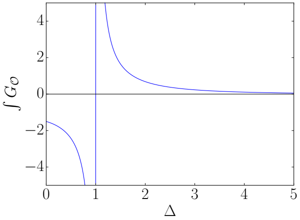
Perturbations to correlators of $\hat{\mathcal{O}}$ itself compute $\beta(g)$.

For exactly marginal operator [Bashmakov, Bertolini, Raj; 1709.01749] [CB; 1709.03967] ,

$$\beta_0 = 0 \Rightarrow \hat{\Delta} = d$$

$$\beta_1 = 0 \Rightarrow \lambda_{\hat{\mathcal{O}}\hat{\mathcal{O}}\hat{\mathcal{O}}} = 0$$

$$\beta_2 = 0 \Rightarrow \sum_{\mathcal{O}} \lambda^2_{\hat{\mathcal{O}}\hat{\mathcal{O}}\mathcal{O}} \int_S G_{\mathcal{O}}(u, v)|x|^{-2d} dx = 0$$



# Perturbation theory for a general beta function

$$\text{“}S\text{“} \mapsto \text{“}S\text{“} + \int g\hat{\mathcal{O}}dx$$

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \mapsto \left\langle \mathcal{O}_1 \dots \mathcal{O}_n e^{\int g\hat{\mathcal{O}}dx} \right\rangle$$

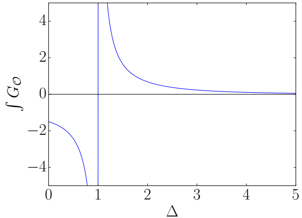
Perturbations to correlators of $\hat{\mathcal{O}}$ itself compute $\beta(g)$.

For exactly marginal operator [Bashmakov, Bertolini, Raj; 1709.01749] [CB; 1709.03967] ,

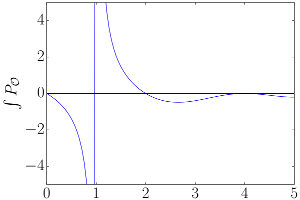
$$\beta_0 = 0 \Rightarrow \hat{\Delta} = d$$

$$\beta_1 = 0 \Rightarrow \lambda_{\hat{\mathcal{O}}\hat{\mathcal{O}}\hat{\mathcal{O}}} = 0$$

$$\beta_2 = 0 \Rightarrow \sum_{\mathcal{O}} \lambda^2_{\hat{\mathcal{O}}\hat{\mathcal{O}}\mathcal{O}} \int_S G_{\mathcal{O}}(u,v)|x|^{-2d}dx = 0$$




Special basis for four-point function in 1D [Mazáč, Paulos; 1811.10646] .



# Conformal manifold as a dynamical system

First-order CPT is enough if $\beta(g) = 0$ [CB; 1709.03967] [Hollands; 1710.05601] .

# Conformal manifold as a dynamical system

First-order CPT is enough if $\beta(g) = 0$ [CB; 1709.03967] [Hollands; 1710.05601] .

$$\delta\Delta_i = -\delta g S_{d-1} \lambda_{ii\hat{\mathcal{O}}} + O(\delta g^2)$$

$$\delta\lambda_{ijk} = \delta g \int \left\langle \mathcal{O}_i(0) \hat{\mathcal{O}}(x) \mathcal{O}_j(\hat{e}) \mathcal{O}_k(\infty) \right\rangle dx + O(\delta g^2)$$

# Conformal manifold as a dynamical system

First-order CPT is enough if $\beta(g) = 0$ [CB; 1709.03967] [Hollands; 1710.05601] .

$$\partial_g \Delta_i = -S_{d-1} \lambda_{ii\hat{\mathcal{O}}}$$

$$\partial_g \lambda_{ijk} = \int \left\langle \mathcal{O}_i(0) \hat{\mathcal{O}}(x) \mathcal{O}_j(\hat{e}) \mathcal{O}_k(\infty) \right\rangle dx$$

# Conformal manifold as a dynamical system

First-order CPT is enough if $\beta(g) = 0$ [CB; 1709.03967] [Hollands; 1710.05601] .

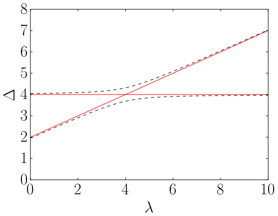
$$
\begin{aligned}
\partial_g \Delta_i &= -S_{d-1} \lambda_{ii\hat{\mathcal{O}}} \\
\partial_g \lambda_{ijk} &= \sum_{\mathcal{O}} \left[ \lambda_{i\hat{\mathcal{O}}\mathcal{O}} \lambda_{jk\mathcal{O}} \int_S \frac{G_{\mathcal{O}}(u,v)}{|x|^{\Delta_i + d}} dx \right. \\
&\qquad + \lambda_{j\hat{\mathcal{O}}\mathcal{O}} \lambda_{ik\mathcal{O}} \int_T \frac{G_{\mathcal{O}}(v,u)}{|x|^{\Delta_j + d}} dx \\
&\qquad \left. + \lambda_{k\hat{\mathcal{O}}\mathcal{O}} \lambda_{ij\mathcal{O}} \int_U \frac{G_{\mathcal{O}}(1/u, v/u)}{|x|^{\Delta_k + d}} dx \right]
\end{aligned}
$$

# Conformal manifold as a dynamical system

First-order CPT is enough if $\beta(g) = 0$ [CB; 1709.03967] [Hollands; 1710.05601] .

$$\partial_g \Delta_i = -S_{d-1} \lambda_{ii\hat{\mathcal{O}}}$$

$$\begin{aligned} \partial_g \lambda_{ijk} = \sum_{\mathcal{O}} \Bigg[ & \lambda_{i\hat{\mathcal{O}}\mathcal{O}} \lambda_{jk\mathcal{O}} \int_S \frac{G_{\mathcal{O}}(u,v)}{|x|^{\Delta_i + d}} dx \\ & + \lambda_{j\hat{\mathcal{O}}\mathcal{O}} \lambda_{ik\mathcal{O}} \int_T \frac{G_{\mathcal{O}}(v,u)}{|x|^{\Delta_j + d}} dx \\ & + \lambda_{k\hat{\mathcal{O}}\mathcal{O}} \lambda_{ij\mathcal{O}} \int_U \frac{G_{\mathcal{O}}(1/u, v/u)}{|x|^{\Delta_k + d}} dx \Bigg] \end{aligned}$$

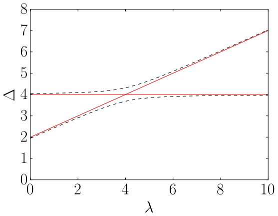


# Conformal manifold as a dynamical system

First-order CPT is enough if $\beta(g) = 0$ [CB; 1709.03967] [Hollands; 1710.05601] .

$$\partial_g \Delta_i = -S_{d-1} \lambda_{ii\hat{\mathcal{O}}}$$

$$\partial_g \lambda_{ijk} = \sum_{\mathcal{O}} \left[ \lambda_{i\hat{\mathcal{O}}\mathcal{O}} \lambda_{jk\mathcal{O}} \int_S \frac{G_{\mathcal{O}}(u,v)}{|x|^{\Delta_i + d}} dx \right.$$
$$+ \lambda_{j\hat{\mathcal{O}}\mathcal{O}} \lambda_{ik\mathcal{O}} \int_T \frac{G_{\mathcal{O}}(v,u)}{|x|^{\Delta_j + d}} dx$$
$$\left. + \lambda_{k\hat{\mathcal{O}}\mathcal{O}} \lambda_{ij\mathcal{O}} \int_U \frac{G_{\mathcal{O}}(1/u, v/u)}{|x|^{\Delta_k + d}} dx \right]$$




Naive crossing in $\mathcal{N} = 4$ SYM [Korchemsky; 1512.05362] :

$$\mathcal{O}_1 = \frac{1}{N} Tr[X^I X_I] \qquad \Delta_1 \sim 2\lambda^{\frac{1}{4}}$$

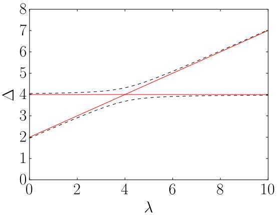
$$\mathcal{O}_2 = \frac{1}{N^2} Tr[X^I X^J] Tr[X_I X_J] \qquad \Delta_2 = 4 + O\left(\frac{1}{N^2}\right)$$

# Conformal manifold as a dynamical system

First-order CPT is enough if $\beta(g) = 0$ [CB; 1709.03967] [Hollands; 1710.05601] .

$$\partial_g \Delta_i = -S_{d-1} \lambda_{ii\hat{\mathcal{O}}}$$

$$\partial_g \lambda_{ijk} = \sum_{\mathcal{O}} \left[ \lambda_{i\hat{\mathcal{O}}\mathcal{O}} \lambda_{jk\mathcal{O}} \int_S \frac{G_{\mathcal{O}}(u,v)}{|x|^{\Delta_i + d}} dx + \lambda_{j\hat{\mathcal{O}}\mathcal{O}} \lambda_{ik\mathcal{O}} \int_T \frac{G_{\mathcal{O}}(v,u)}{|x|^{\Delta_j + d}} dx + \lambda_{k\hat{\mathcal{O}}\mathcal{O}} \lambda_{ij\mathcal{O}} \int_U \frac{G_{\mathcal{O}}(1/u, v/u)}{|x|^{\Delta_k + d}} dx \right]$$




Naive crossing in $\mathcal{N} = 4$ SYM [Korchemsky; 1512.05362] :

$$\mathcal{O}_1 = \frac{1}{N} Tr[X^I X_I] \qquad \Delta_1 \sim 2\lambda^{\frac{1}{4}}$$

$$\mathcal{O}_2 = \frac{1}{N^2} Tr[X^I X^J] Tr[X_I X_J] \qquad \Delta_2 = 4 + O\left(\frac{1}{N^2}\right)$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \frac{\gamma}{N} \qquad \Delta_\pm = \frac{\Delta_1 + \Delta_2}{2} \pm \sqrt{\frac{\Delta_{12}^2}{4} + \frac{\gamma^2}{N^2}}$$

# Conclusion

- The bootstrap forces us to consider nonlocal and even nonunitary theories.
- Fermionic fixed-points with exceptional or discrete symmetry remain largely unexplored.
- Patterns in the spectrum can help us locate a theory even without a kink.
- Shadow (superblock) construction works for many CFTs.
- Simplest conformal manifolds tractable with ODE system.
- Generalization to include curvature and spinning blocks will be important for the future.