Conformal invariance: What is it good for?

Connor Behan

March 22, 2018

Types of QFTs to study

Fundamental theories

* The Standard Model
* QED, QCD, etc
* Inflationary models

Collective phenomena

* The Ising model
* Topological QFTs, anyons
* Quantum Hall effect

"Beautiful" theories

* N=4 Super Yang-Mills
* (Seiberg / Donaldson)-Witten
* ABJM theory

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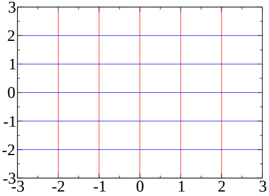

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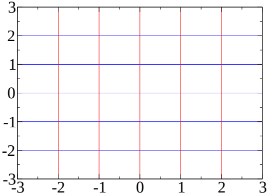
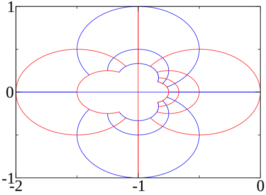


$$\xrightarrow{b_\mu=(1,0)}$$



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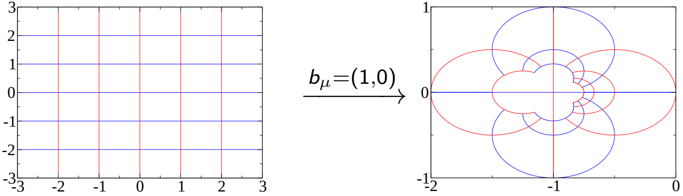

Nonlinear transformation is really $K_\mu = I \circ P_\mu \circ I$.

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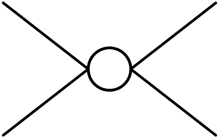

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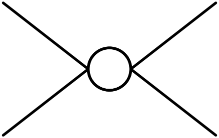

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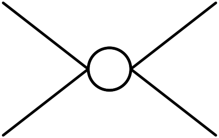

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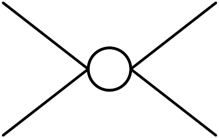

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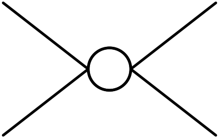

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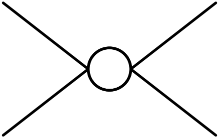

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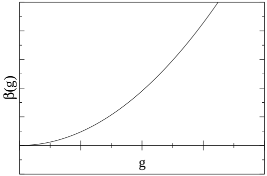

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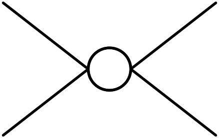

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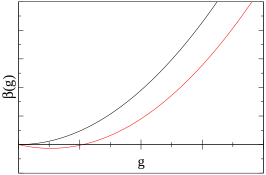

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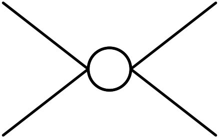

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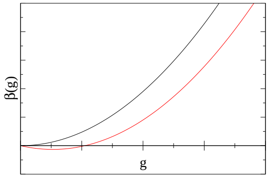

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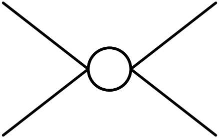

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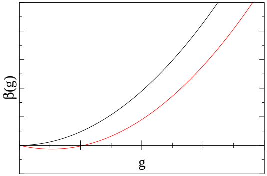

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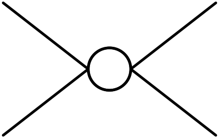

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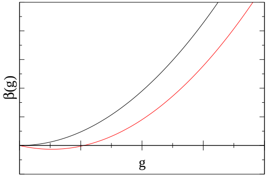

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Outline

- Interesting things about conformal invariance
    - It is one of the few enhanced symmetries
    - Second order phase transitions are conformal
    - Some 2D CFTs directly describe string theory
    - Higher dimensional CFTs arise indirectly in string theory
- Infinitely many 2D CFTs are exactly solvable
- The past decade has been filled with new techniques
    - Most precise values for 3D Ising critical exponents
    - Interplay with supersymmetric dualities
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- Another way out is to use superalgebras.
- SUSY is $\{Q_i, Q_j\} = 2\delta_{ij}\gamma^\mu P_\mu$ where $i, j = 1, \dots, \mathcal{N}$.
- One usually stops at 3D $\mathcal{N} = 8$ or 4D $\mathcal{N} = 4$.
- Most known CFTs are SCFTs — superconformal.

Phase transitions

Lattice Ising model:

Leading term in $\langle \sigma(x)\sigma(0) \rangle$ is $e^{-|x|/\xi}$ above T_c or $\frac{1}{|x|^{2\Delta_\sigma}}$ at T_c.

Adding $t\xi^{\Delta_\epsilon - d} \sum_i \epsilon_i$ for $t < 0$ will align spins.

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Propagator $\langle \phi(x)\phi(0) \rangle$ is either $\frac{e^{-m|x|}}{|x|}$ in 3D or $\frac{mK_1(m|x|)}{|x|}$ in 4D.

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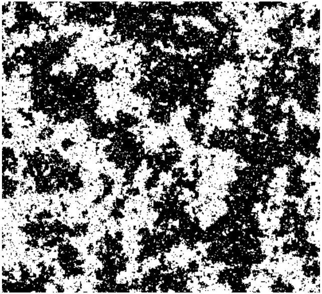

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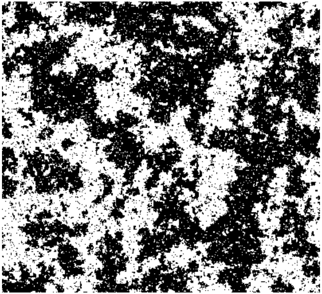

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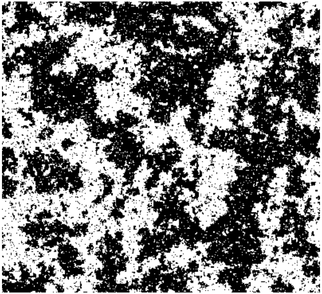

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When there is no typical cluster size, we have scale and usually conformal invariance.

Worldlines and worldsheets

$$S = \frac{m}{2} \int_{\mathbb{R}} \delta^{ij} \dot{x}_i \dot{x}_j dt$$

0+1D QFT with d scalar fields

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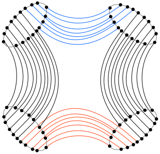

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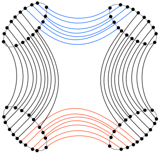

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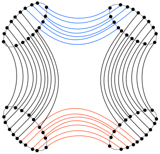

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Anti-de Sitter space

Solving Einstein's equation for negative $\Lambda = -\frac{d(d-1)}{2L^2}$ yields

$$ds^2 \;=\; -\left(1+\frac{r^2}{L^2}\right)dt^2+\left(1+\frac{r^2}{L^2}\right)^{-1}dr^2+r^2 d\Omega_{d-1}^2$$

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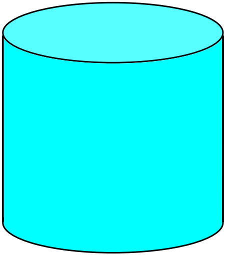

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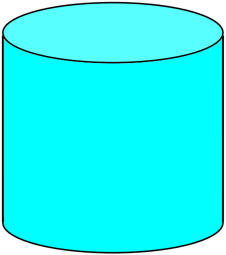

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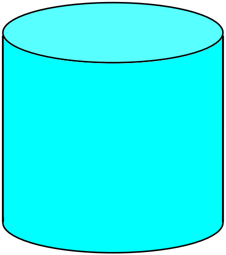

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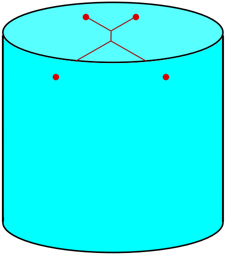

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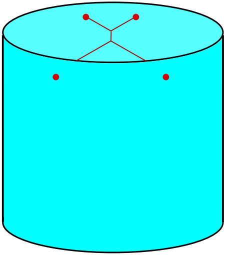

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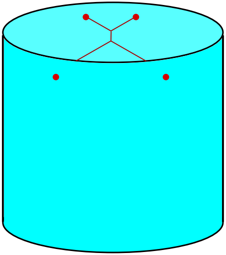

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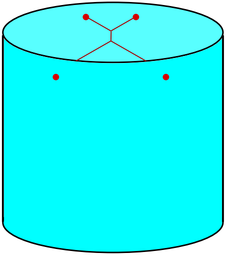

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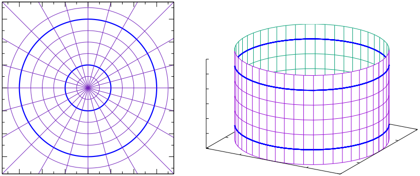
$\mathcal{N} = 4$ Super Yang-Mills		λ	A_μ		φ	ψ
	Φ	$\tilde{\lambda}$		$\tilde{\psi}^\dagger$	$\tilde{\varphi}^\dagger$	

Outline

- Interesting things about conformal invariance
  - It is one of the few enhanced symmetries
  - Second order phase transitions are conformal
  - Some 2D CFTs directly describe string theory
  - Higher dimensional CFTs arise indirectly in string theory
- Infinitely many 2D CFTs are exactly solvable
- The past decade has been filled with new techniques
  - Most precise values for 3D Ising critical exponents
  - Interplay with supersymmetric dualities
  - Numerical bounds on all CFTs from Euclidean space
  - Analytic bounds on all CFTs from Minkowski space
  - Extensions to non-conformal theories

The Hilbert space



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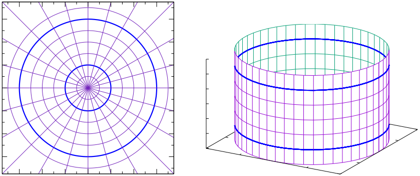

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Primary operator: $[K_\mu \mathcal{O}(0)] = 0$
Descendants:
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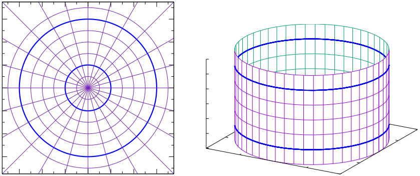

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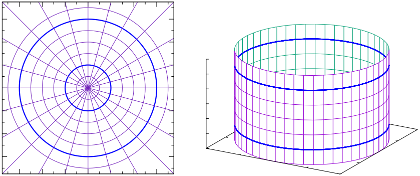

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Operator product expansion (OPE):

$$\phi(x)\phi(0) = \sum_{\mathcal{O}} \frac{\lambda_{12\mathcal{O}}}{|x|^{\Delta_1 + \Delta_2 - \Delta}} C(x, \partial)\mathcal{O}(0)$$

More conformal invariance in 2D

If $z, \bar{z} = x \pm iy$,

$$ds^2 = dx^2 + dy^2 = dz d\bar{z} \mapsto dz' d\bar{z}' = \frac{dz'}{dz} \frac{d\bar{z}'}{d\bar{z}} dz d\bar{z}$$

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$$\begin{array}{lll} |h\rangle & & \\ L_{-1}|h\rangle & & \\ L_{-1}^2|h\rangle & L_{-2}|h\rangle & \\ L_{-1}^3|h\rangle & L_{-1}L_{-2}|h\rangle & L_{-3}|h\rangle \end{array}$$

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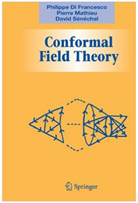

Unitary 2D CFTs

We need the determinant of $M = diag(M^{(0)}, M^{(1)}, M^{(2)}, \dots)$.

$$\begin{vmatrix} \langle h|h\rangle & 0 & 0 & 0 & \dots \\ 0 & \langle h|L_1L_{-1}|h\rangle & 0 & 0 & \dots \\ 0 & 0 & \left\langle h|L_1^2L_{-1}^2|h\right\rangle & \left\langle h|L_2L_{-1}^2|h\right\rangle & \dots \\ 0 & 0 & \left\langle h|L_1^2L_{-2}|h\ \right\rangle & \left\langle h|L_2L_{-2}|h\right\rangle & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

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Unitary 2D CFTs

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Comes from a search for singular vectors like

$$\sum_{p_1+\dots+p_k=s} \frac{[(s-1)!]^2[c-13+\sqrt{(1-c)(25-c)}]^{s-k}}{\prod_{i=1}^{k-1}(p_1+\ldots p_i)(s-p_1-\dots-p_i)} L_{-p_1} \dots L_{-p_k} |h_{1,s}\rangle$$

Unitary 2D CFTs

$$c = 1 - \frac{6}{m(m+1)}$$

$$h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}$$

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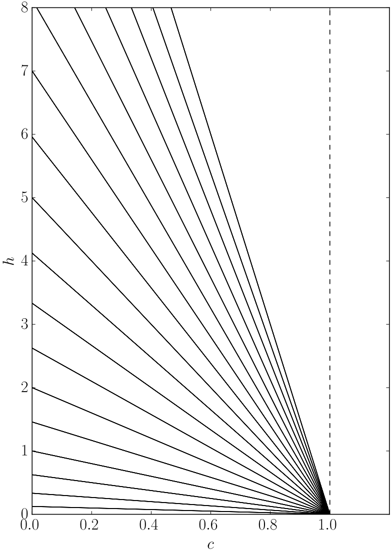

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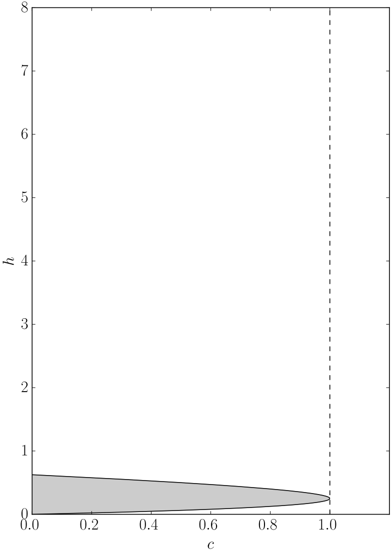

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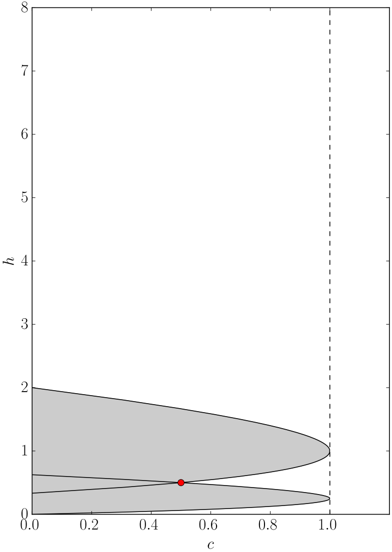

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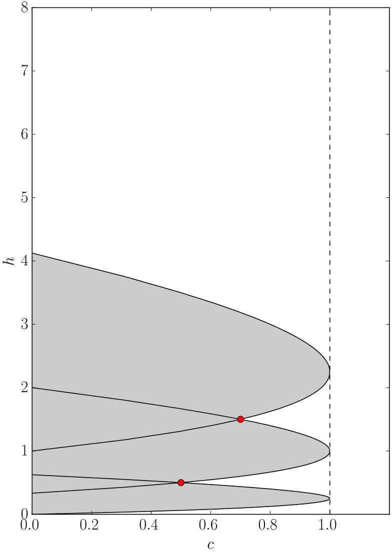

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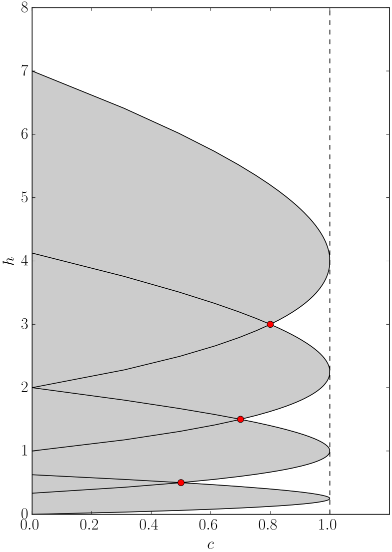

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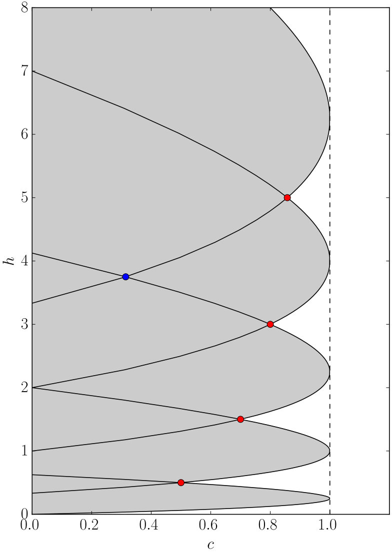

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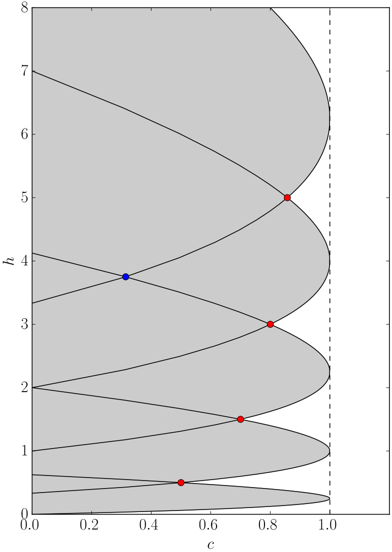

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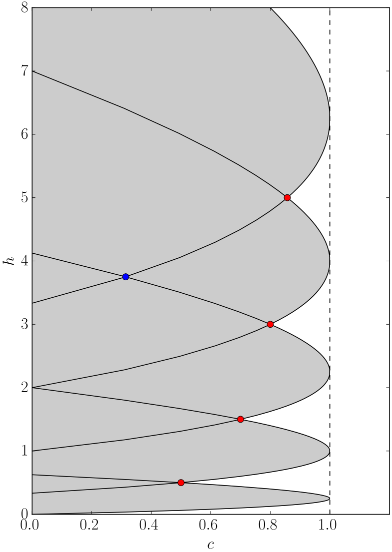
Additional reflection symmetry:

$$h_{r,s} = h_{m-r,m+1-s}$$



Minimal models

Name	m	c	Operators
Ising	3	$\frac{1}{2}$	$\phi_{(2,1)}$ $\phi_{(2,2)}$ $\phi_{(2,3)}$ $\phi_{(1,1)}$ $\phi_{(1,2)}$ $\phi_{(1,3)}$
Tricritical Ising	4	$\frac{7}{10}$	$\phi_{(3,1)}$ $\phi_{(3,2)}$ $\phi_{(3,3)}$ $\phi_{(3,4)}$ $\phi_{(2,1)}$ $\phi_{(2,2)}$ $\phi_{(2,3)}$ $\phi_{(2,4)}$ $\phi_{(1,1)}$ $\phi_{(1,2)}$ $\phi_{(1,3)}$ $\phi_{(1,4)}$
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Primary ↦ descendant OPE coefficients:

$$\langle L_{-n}\mathcal{O}(z)\mathcal{O}_1(z_1)\dots\mathcal{O}_n(z_n)\rangle = \mathcal{L}_{-n}\langle\mathcal{O}(z)\mathcal{O}_1(z_1)\dots\mathcal{O}_n(z_n)\rangle$$

$$\mathcal{L}_{-n} = \sum_{i=1}^{n}\left[\frac{h_i(n-1)}{(z_i-z)^n} - \frac{\partial_i}{(z_i-z)^{n-1}}\right]$$

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Primaries fuse according to:

$$\phi_{(r,s)} \times \phi_{(p,q)} = \sum_{k=|p-r|+1}^{min(p+r-1,2m-1-p-r)} \sum_{l=|q-s|+1}^{min(q+s-1,2m+1-q-s)} \phi_{(k,l)}$$

Minimal models

Study four-point function with $\phi_{(r,s)}$ twice and $\phi_{(p,q)}$ twice.

$$\left\langle P_{rs}(L)\phi_{(r,s)}\phi_{(p,q)}\phi_{(r,s)}\phi_{(p,q)}\right\rangle = 0$$

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For $\langle\sigma(0)\sigma(z)\sigma(1)\sigma(\infty)\rangle = \frac{G(z)}{z^{2h_\sigma}}$, ODE is:

$$\frac{3}{2}z(z-1)^2\frac{d^2G}{dz^2} + \frac{1}{8}(z-1)(15z-6)\frac{dG}{dz} - \frac{9}{128}zG = 0$$

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Fix $\lambda_{\sigma\sigma\epsilon} = \frac{1}{2}$ using $z \leftrightarrow 1-z$ symmetry and:

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1\left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix} \middle| 1-z\right)$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a, c-b \\ c-a-b+1 \end{matrix} \middle| 1-z\right)$$

Outline

- Interesting things about conformal invariance
  - It is one of the few enhanced symmetries
  - Second order phase transitions are conformal
  - Some 2D CFTs directly describe string theory
  - Higher dimensional CFTs arise indirectly in string theory
- Infinitely many 2D CFTs are exactly solvable
- The past decade has been filled with new techniques
  - Most precise values for 3D Ising critical exponents
  - Interplay with supersymmetric dualities
  - Numerical bounds on all CFTs from Euclidean space
  - Analytic bounds on all CFTs from Minkowski space
  - Extensions to non-conformal theories

The modern bootstrap

$$\langle \phi_i(x_1)\phi_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{\Delta_i+\Delta_j}}$$

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle = \frac{\lambda_{ijk}}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{13}|^{\Delta_i+\Delta_k-\Delta_j}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}}$$

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Cross-ratios are $u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.

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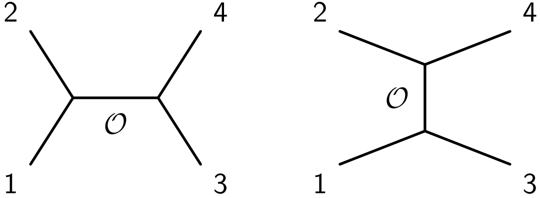
Virasoro blocks not available but we have: [Dolan, Osborn; 0011040, 0309180]

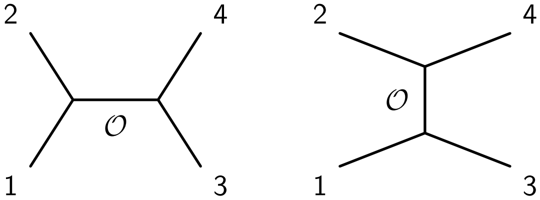
$$\begin{aligned} g_{\Delta,\ell}(z,\bar{z}) &= z^h \bar{z}^{\bar{h}} {}_2F_1\left(\begin{matrix} h-\Delta_{12}, h+\Delta_{34} \\ 2h \end{matrix} \Big| z\right) {}_2F_1\left(\begin{matrix} \bar{h}-\Delta_{12}, \bar{h}+\Delta_{34} \\ 2\bar{h} \end{matrix} \Big| \bar{z}\right) \\ &\quad + (z \leftrightarrow \bar{z}) \end{aligned}$$

The modern bootstrap



Two ways to compute $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ by switching $1 \leftrightarrow 3$.

The modern bootstrap



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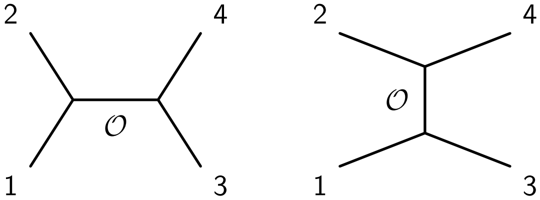

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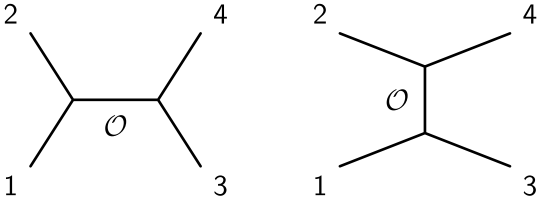

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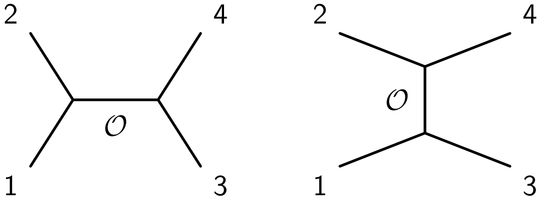

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What if some functional α is positive for every block in a trial spectrum? [Rattazi, Rychkov, Tonni, Vichi; 0807.0004]

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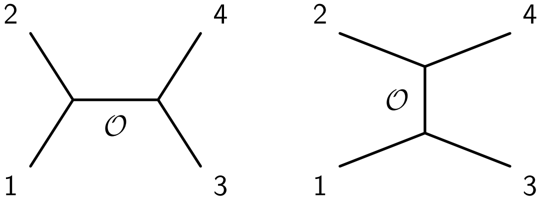

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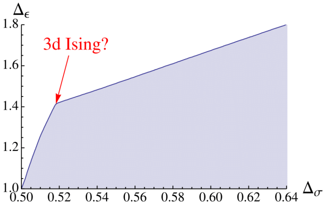
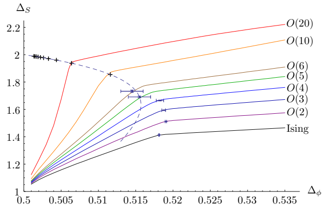
Free software: [Paulos; 1412.4127] [Simmons-Duffin; 1502.02033] [CB; 1602.02810]

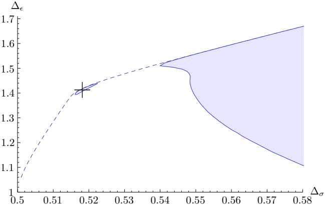
Flagship results



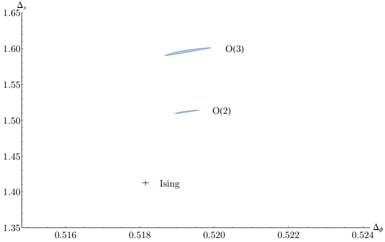
[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi; 1203.6064]



[Kos, Poland, Simmons-Duffin; 1307.6856]



[Kos, Poland, Simmons-Duffin; 1406.4858]



[Kos, Poland, Simmons-Duffin, Vichi; 1603.04436]

Superconformal blocks

Superprimary (killed by S) $\Rightarrow$ primary (killed by K) but not v/v.

$$\left\{ Q_\alpha, \bar{Q}_{\dot\alpha} \right\} = \sigma^\mu_{\alpha\dot\alpha} P_\mu \ , \ \left\{ S_\alpha, \bar{S}_{\dot\alpha} \right\} = \sigma^\mu_{\alpha\dot\alpha} K_\mu$$

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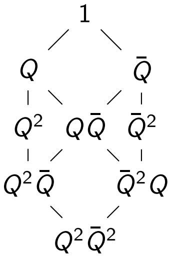

1

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Q^2 $Q\bar{Q}$ $\bar{Q}^2$

$Q^2\bar{Q}$ $\bar{Q}^2 Q$

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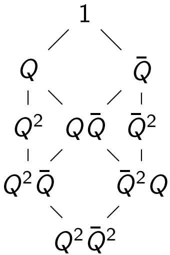

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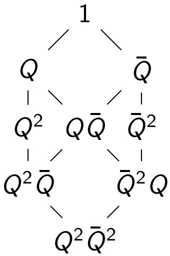
$$\begin{aligned} G_{\Delta,\ell} \quad &= \quad g_{\Delta,\ell} + \frac{(\Delta+\ell)(\Delta-\ell-2)}{(\Delta+\ell+1)(\Delta-\ell-1)} g_{\Delta+2,\ell} - \\ &\quad \frac{(\ell+2)(\Delta+\ell)}{(\ell+1)(\Delta+\ell+1)} g_{\Delta+1,\ell+1} - \frac{\ell(\Delta-\ell-2)}{(\ell+1)(\Delta-\ell-1)} g_{\Delta+1,\ell-1} \end{aligned}$$

Generalized in: [Bobev, El-Showk, Mazac, Paulos; 1503.02081]



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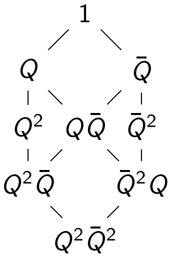

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Many multiplets to consider:

[Fitzpatrick, Poland, Khandker, Li, Poland, Simmons-Duffin; 1402.1167]

[Khandker, Li, Poland, Simmons-Duffin; 1404.5300]

Recent numerics with above blocks:

[Poland, Stergiou; 1509.06368]

[Li, Meltzer, Stergiou; 1702.00404]

Extended supersymmetry

In most superconformal algebras, $T_{\mu\nu}$ is not a primary:

$$\mathcal{O} = Tr[\phi^I \phi^J]$$

$$\dots$$

$$\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} Q^3_\alpha Q^3_\beta Q^4_\gamma Q^4_\delta \mathcal{O} \quad \sigma^{\alpha\dot{\alpha}}_\mu \sigma^{\beta\dot{\beta}}_\nu \bar{Q}^1_{\dot{\alpha}} \bar{Q}^2_{\dot{\beta}} Q^3_\alpha Q^4_\beta \mathcal{O} \quad \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\gamma}\dot{\delta}} \bar{Q}^1_{\dot{\alpha}} \bar{Q}^1_{\dot{\beta}} \bar{Q}^2_{\dot{\gamma}} \bar{Q}^2_{\dot{\delta}}$$

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Calculate protected part, bootstrap unprotected part:

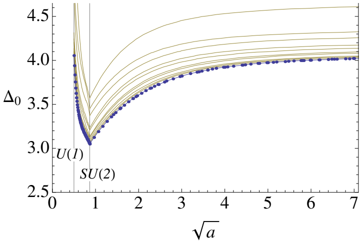
$$\langle \mathcal{O}(x_1, w_1)\mathcal{O}(x_2, w_2)\mathcal{O}(x_3, w_3)\mathcal{O}(x_4, w_4)\rangle = free + \frac{F(z,\bar{z})}{x_{13}^2 x_{24}^2}$$

$$\left[\frac{(w_{12}w_{34})^2}{(x_{12}x_{34})^2} + \frac{w_{12}w_{23}w_{34}w_{41}}{(x_{12}x_{23}x_{34}x_{41})^2}(x_{13}^2x_{24}^2 - x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2) + perms\right]$$

$$F(z,\bar{z}) = \frac{(z+\bar{z}-1)(2|z|^2 - z - \bar{z} + 1) - |z|^4 - 1}{|z|^2|1-z|^2} - \frac{1}{c|z|^2}$$

$$+ \left|\frac{1-z}{z}\right|^2 G(z,\bar{z})$$

Extended supersymmetry



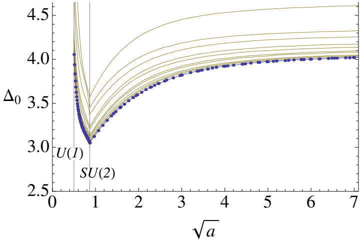
[Beem, Rastelli, van Rees; 1304.1803]

Extended supersymmetry

Restrict ops to
$x^{\mu} = \left(0, 0, \frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$.
Let $\mathcal{Q} = Q_1 + \bar{S}_2 = (S_1 + \bar{Q}_2)^{\dagger}$.
Let $\mathcal{O}_i$ be killed by $\mathcal{Q}$.

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(0) = \sum_k \frac{\lambda_{ijk}}{z^{h_i+h_j-h_k}} \mathcal{O}_k(0)$$

True up to $\{\mathcal{Q}, \ldots]$.



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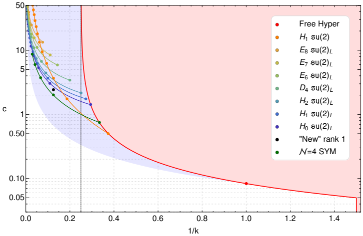
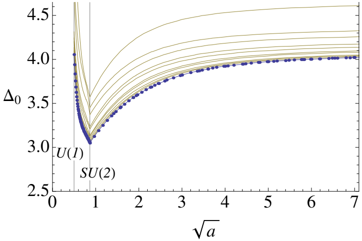


[Beem, Lemos, Liendo, Rastelli, van Rees; 1412.7541]



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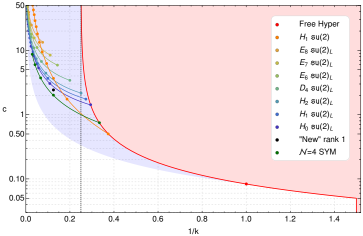
2D-4D correspondence

[Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees; 1312.5344]

1D-3D correspondence

[Dedushenko, Pufu, Yavoby; 1610.00740]



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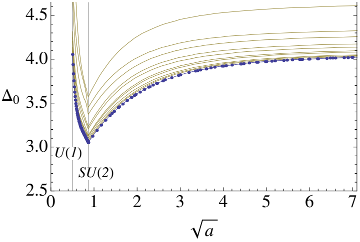

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Exploit conformal group being $SO(d+1,1)$.

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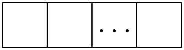
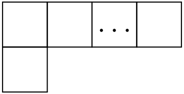
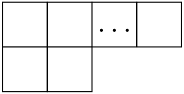


$$\langle \phi J_\mu \phi J_\nu \rangle \quad \sim \quad above +$$



$$\langle \phi T_{\mu\nu} \phi T_{\rho\sigma} \rangle \quad \sim \quad above +$$



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Exploit conformal group being $SO(d+1,1)$.

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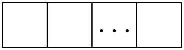

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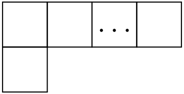

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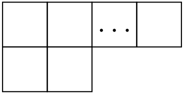

Use $(X^+, X^-, X^\mu) = (1, x^2, x^\mu), (W^+, W^-, W^\mu) = (0, w\cdot x, w^\mu)$ instead of points x^μ, w^μ. [Costa, Penedones, Poland, Rychkov; 1107.3554, 1109.6321]

$$\begin{tikzcd}
X^A \arrow[r, "Lorentz"] & X'^A \arrow[d] \\
x^\mu \arrow[u] \arrow[r, "Conformal"] & x'^\mu
\end{tikzcd}$$

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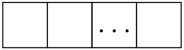

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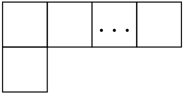

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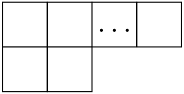

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Operators satisfy $\Phi(\alpha X, \beta W) = \alpha^{-\Delta}\beta^{\ell}\Phi(X, W)$.

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Symmetric n_{ij} such that $m_i \equiv \ell_i - \sum_{j\neq i} n_{ij} \geq 0$ characterize all three-point structures.

$$\begin{aligned} H_{ij} &= -2[W_{ij}X_{ij} - (W_i \cdot X_j)(W_j \cdot X_i)] \\ V_{i,jk} &= \frac{(W_i \cdot X_j)X_{ik} - (W_i \cdot X_k)X_{ij}}{X_{jk}} \end{aligned}$$

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Stress-tensor example:

$$\langle T(X_1, W_1) T(X_2, W_2) T(X_3, W_3) \rangle = \frac{P}{X_{12}^{\frac{d+2}{2}} X_{13}^{\frac{d+2}{2}} X_{23}^{\frac{d+2}{2}}}$$

$$\begin{aligned} P = {} & x_4 V_1^2 V_2^2 V_3^2 + x_3(V_1^2 H_{23}^2 + \dots) + x_2(V_2 V_3 H_{12} H_{13} + \dots) \\ & + x_1(V_1^2 V_2 V_3 H_{23} + \dots) + x_0 H_{12} H_{13} H_{23} \end{aligned}$$

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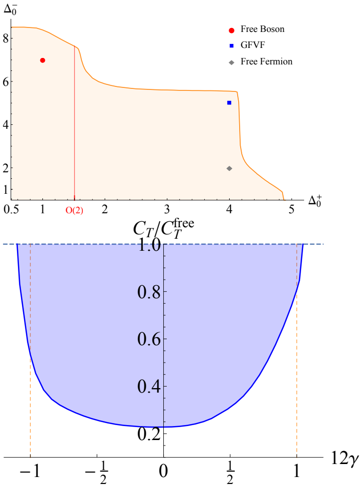
Further developments made in:

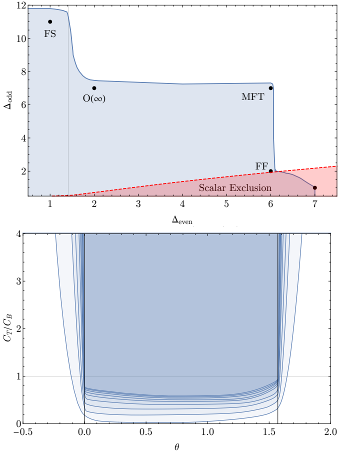
[Echeverri, Elkhidir, Karateev, Serone; 1505.03750, 1601.05325] [Cuomo, Karateev, Kravchuk; 1705.05401]

[Costa, Hansen, Penedones, Trevisani; 1603.05551, 1603.05552] [Karateev, Kravchuk, Simmons-Duffin; 1706.07813]

Current and stress-tensor bootstraps



[Dymarsky, Penedones, Trevisani, Vichi; 1705.04278]



[Dymarsky, Kos, Kravchuk, Poland, Simmons-Duffin; 1708.05718]

Central charge bounds

$\langle TTT \rangle$ has bosonic and fermionic parts

$$\langle TTT \rangle_{d=3} \quad = \quad \frac{C_T \cos\theta}{\sin\theta + \cos\theta} \langle TTT \rangle_{scalar} + \frac{C_T \sin\theta}{\sin\theta + \cos\theta} \langle TTT \rangle_{spinor}$$

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Positive by average null energy condition: [Hoffman, Maldacena; 0803.1467]

$$\langle \psi | \varepsilon(\theta) | \psi \rangle \geq 0 \; , \; \varepsilon(\theta) = \lim_{r\to\infty} r^{d-2} \int_{-\infty}^{\infty} T^0_i n^i dt$$

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Positive by average null energy condition: [Hoffman, Maldacena; 0803.1467]

$$\langle \psi | \varepsilon(\theta) | \psi \rangle \geq 0 \, , \; \varepsilon(\theta) = \lim_{r \to \infty} r^{d-2} \int_{-\infty}^{\infty} T^0_i n^i \, dt$$

Coefficients are related to the Weyl anomaly

$$\begin{aligned} \langle T^\mu_\mu \rangle_{d=2} &= -\frac{c}{12} R \\ \langle T^\mu_\mu \rangle_{d=4} &= -\frac{a}{16\pi^2} \left(R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4 R^{\alpha\beta} R_{\alpha\beta} + R^2 \right) \\ &- \frac{c}{16\pi^2} \left(R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 2 R^{\alpha\beta} R_{\alpha\beta} + \frac{1}{3} R^2 \right) \end{aligned}$$

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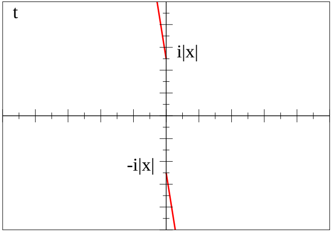
$$\begin{aligned} \frac{1}{(x^2+\tau^2)^\Delta} &= \frac{1}{(x^2+t^2e^{2i\theta})^\Delta} \\ &= \frac{e^{-2i\Delta\theta}}{(t^2+e^{-2i\theta}x^2)^\Delta} \\ &= \frac{e^{\pm\pi i\Delta}}{(t^2-x^2)^\Delta} \end{aligned}$$



Beyond CFTs

For a relevant operator $\mathcal{O}$ in a CFT,

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Better to truncate and diagonalize H_{int} after expressing it in the basis:

$$\{ |\mathcal{O}\rangle \, , P_\mu |\mathcal{O}\rangle \, , P_\mu P_\nu |\mathcal{O}\rangle \, , \dots \}$$

[Hogervorst, Rychkov, van Rees; 1409.1581] [Katz, Khandker, Walters; 1604.01766]

[Elias-Miro, Rychkov, Vitale; 1706.09929]

Scattering amplitudes

AdS with massive particles has a flat space limit:

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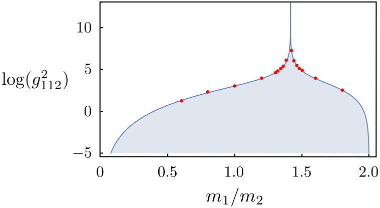
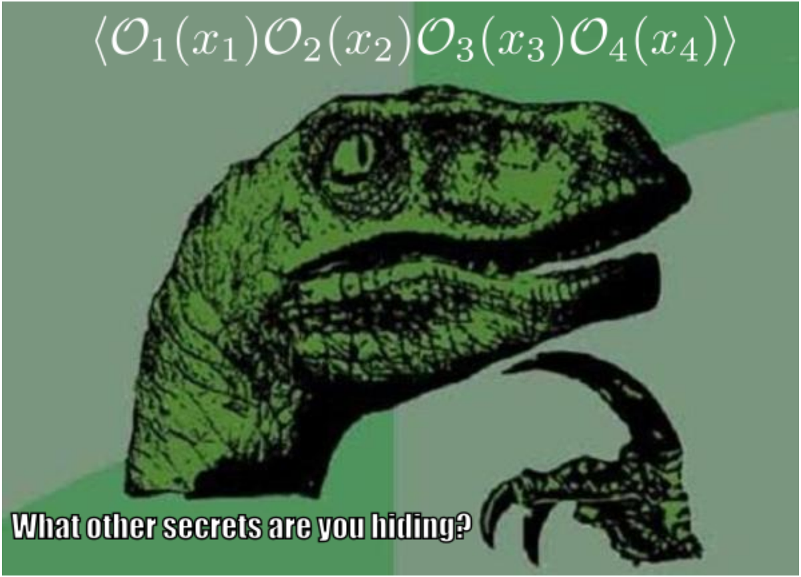


Saturated by Sine-Gordon:

$$S(s) = sgn(m_2^2 - 2m_1^2) \frac{\sqrt{s}\sqrt{4m_1^2 - s} + m_2\sqrt{4m_1^2 - m_2^2}}{\sqrt{s}\sqrt{4m_1^2 - s} - m_2\sqrt{4m_1^2 - m_2^2}}$$


$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$
What other secrets are you hiding?