Bootstrapping the Long-Range Ising Model

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1810.07199
1703.03430, 1703.05325 with L. Rastelli, S. Rychkov, B. Zan

The model

$$H_{LRI} = -J \sum_{i \neq j} \frac{\sigma_i \sigma_j}{|i-j|^{d+s}}$$

- Second-order phase transition in $1 \leq d < 4$ [Dyson; 69] .
- Possible to study with a ϕ^4 interaction [Fisher, Ma, Nickel; 72] .
- Critical exponents depend non-trivially on s for $\frac{d}{2} < s < s_*$ [Sak; 73] .
- MC estimates in 1D and 2D [Angelini, Parisi, Ricci-Tersenghi; 1401.6805] .
- Fixed point is conformal [Paulos, Rychkov, van Rees, Zan; 1509.00008] .

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RG approaches to this critical point involve a fractional derivative:
$\partial^\alpha \mathcal{O}(x) = \int \frac{\mathcal{O}(y)}{|x-y|^{d+\alpha}} dy$ in position space,
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This is the shadow transform if $\alpha = d - 2\Delta$.

Continuum description

$$S_1[\phi] = \int \frac{1}{2}\phi\partial^s\phi + \frac{\lambda}{4!}\phi^4 dx$$

- Local operator spectrum has no stress tensor.
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	$s = \frac{d}{2}$	$s = s_*$
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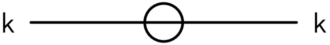
Protected operators



$$= \frac{\lambda^2}{(4\pi)^d} \frac{\Gamma\left(\frac{3\varepsilon - d}{4}\right)\Gamma\left(\frac{d-\varepsilon}{4}\right)^3}{\Gamma\left(\frac{3d-3\varepsilon}{4}\right)\Gamma\left(\frac{d+\varepsilon}{4}\right)^3} |k|^{\frac{d}{2}} + \dots$$

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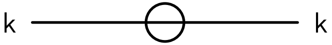

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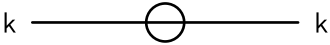

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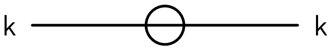

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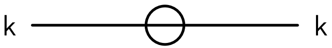

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Introduce a mean-field χ at the short-range end to represent ϕ^3. $\Delta_\sigma \sigma \partial_\mu \chi - \Delta_\chi \chi \partial_\mu \sigma$ will then represent $\partial^\nu T_{\mu\nu}$.

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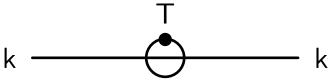

$$= \quad k_{(\mu}k_{\nu)}\frac{\lambda^2}{(4\pi)^d}\frac{d+2}{2\Gamma\left(\frac{d}{4}+1\right)\Gamma\left(\frac{d}{4}+2\right)}\frac{\Gamma\left(\varepsilon\right)\Gamma\left(\frac{\varepsilon}{2}+2\right)\Gamma\left(\frac{d-\varepsilon}{4}\right)^2}{\Gamma\left(\frac{d-\varepsilon}{2}\right)\Gamma\left(\frac{d}{2}+\varepsilon+2\right)\Gamma\left(\frac{d+\varepsilon}{4}\right)^2}+\dots$$

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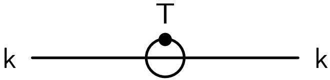

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Dimension at two loops, [CB; 1810.07199]

$$\Delta_T = \frac{d+4-\varepsilon}{2} - \frac{8}{d(d+2)}\left(\frac{\varepsilon}{3}\right)^2 + O(\varepsilon^3)$$

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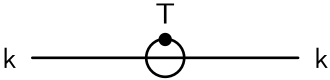

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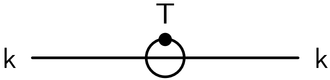

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Dual expressions in 3D using bootstrap data, [CB, Rastelli, Rychkov, Zan; 1703.03430]

$$\begin{aligned}\Delta_T &= 3 + 2.33\delta + O(\delta^2) \\ \Delta_\epsilon &= \Delta_\epsilon^{SRI} + 0.27\delta + O(\delta^2)\end{aligned}$$

Aside

In free theory, $\phi \times \phi$ includes only $[\phi\phi]_{n,\ell} \sim \phi\partial_{\mu_1} \ldots \partial_{\mu_\ell}\partial^{2n}\phi$.

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Consider trajectories in $G(z, \bar{z}) = |z|^{2\Delta_\phi} \langle \phi(0)\phi(z, \bar{z})\phi(1)\phi(\infty) \rangle$.

$$G(z, \bar{z}) \quad = \quad 1 + \sum_{n,\ell} a^{(0)}_{n,\ell} G_{\Delta_{n,\ell},\ell}(z, \bar{z})$$

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$$G(z,\bar{z}) = 1 + \sum_{n,\ell} \left(a^{(0)}_{n,\ell} + a^{(1)}_{n,\ell} + \dots \right) G_{\Delta_{n,\ell} + \gamma^{(1)}_{n,\ell} + \dots, \ell}(z,\bar{z})$$

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Crossed version must give double-log in $\sum_{n,\ell} \gamma^{(2)}_{n,\ell} a^{(0)}_{n,\ell} G'_{\Delta_{n,\ell},\ell}(z,\bar{z})$.

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A double-log is also a double-discontinuity defined by

$$dDisc\,[f(z,\bar{z})] = f(z,\bar{z}) - \frac{1}{2}\left[f(z, e^{2\pi i}\bar{z}) + f(z, e^{-2\pi i}\bar{z})\right]$$

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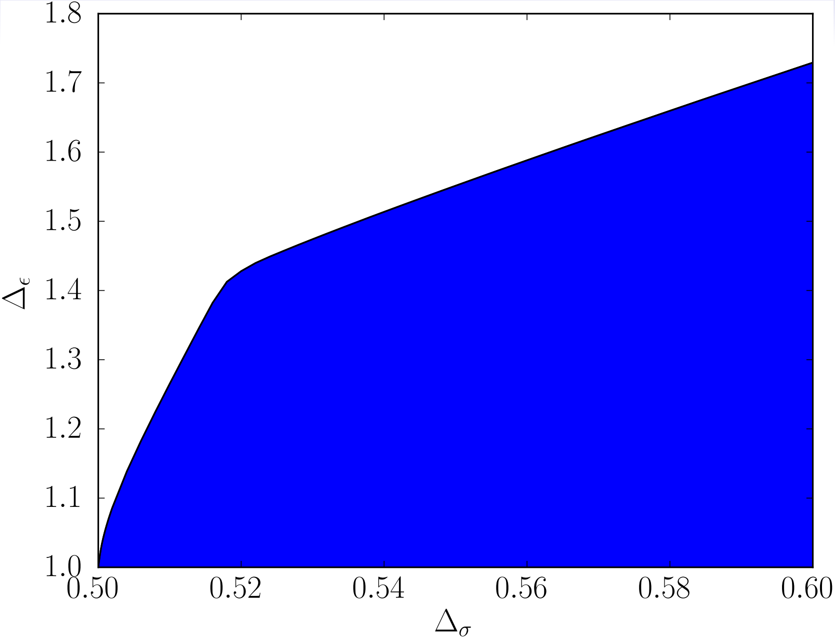
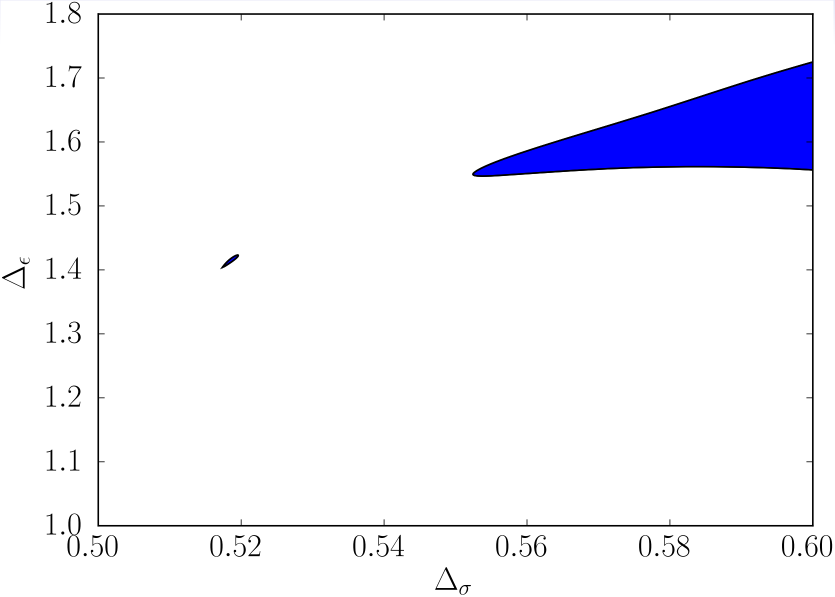
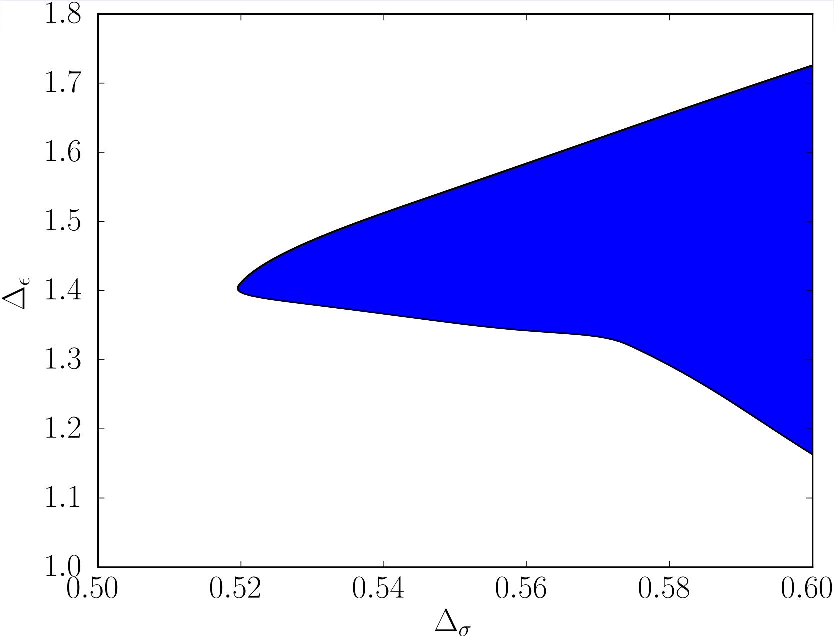
$$\begin{aligned} c(\Delta,\ell) &= \frac{\Gamma\left(\frac{\Delta+\ell}{2}\right)^4}{4\pi^2\Gamma(\Delta+\ell)\Gamma(\Delta+\ell-1)} \\ &\quad \int_0^1\int_0^1 G_{\ell+d-1,\Delta+1-d}(z,\bar{z})dDisc\left[G(z,\bar{z})\right]\frac{\left|z-\bar{z}\right|^{d-2}}{(z\bar{z})^d}dzd\bar{z} \end{aligned}$$



Kink from $\langle\sigma\sigma\sigma\sigma\rangle$: [El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi; 1203.6064] .



Kink from $\langle\sigma\sigma\sigma\sigma\rangle$: [El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi; 1203.6064] .
Island from $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$: [Kos, Poland, Simmons-Duffin; 1406.4858] .


1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
Δ_ϵ
0.50
0.52
0.54
0.56
0.58
0.60
Δ_σ

Bootstrapping a four-point function

Identical scalar case:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{G(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}$$

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$$G(u,v) \;=\; \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} G_{\mathcal{O}}(u,v)$$

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$$\begin{aligned} G(u,v) &= \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 G_{\mathcal{O}}(u,v) \\ &= \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 \left(\frac{u}{v}\right)^{\Delta_\phi} G_{\mathcal{O}}(v,u) \end{aligned}$$

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Crossing equation with $\lambda^2_{\phi\phi\mathcal{O}} \geq 0$ by unitarity:

$$\sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} \left[v^{\Delta_\phi} G_{\mathcal{O}}(u,v) - u^{\Delta_\phi} G_{\mathcal{O}}(v,u) \right] = 0$$

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Find positive functional [Rattazzi, Rychkov, Tonni, Vichi; 0807.0004] !

Bootstrapping many four-point functions

If we define

$$F^{ij;kl}_{\pm,\mathcal{O}} = v^{\frac{\Delta_k+\Delta_j}{2}} G^{\Delta_{ij},\Delta_{kl}}_{\mathcal{O}}(u,v) \pm u^{\frac{\Delta_k+\Delta_j}{2}} G^{\Delta_{ij},\Delta_{kl}}_{\mathcal{O}}(v,u)\ ,$$

the single correlator crossing equation is

$$\sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} F^{\phi\phi;\phi\phi}_{-,\mathcal{O}}(u,v) = 0$$

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Mixed correlator like $\langle \phi\phi\Phi\Phi \rangle$ gives three equations:

$$\sum_{\mathcal{O}} \lambda^2_{\phi\Phi\mathcal{O}} F^{\phi\Phi;\phi\Phi}_{-,\mathcal{O}}(u,v) = 0$$

$$\sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}\lambda_{\Phi\Phi\mathcal{O}} F^{\phi\phi;\Phi\Phi}_{-,\mathcal{O}}(u,v) + \sum_{\mathcal{O}} (-1)^{\ell} \lambda^2_{\phi\Phi\mathcal{O}} F^{\Phi\phi;\phi\Phi}_{-,\mathcal{O}}(u,v) = 0$$

$$\sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}\lambda_{\Phi\Phi\mathcal{O}} F^{\phi\phi;\Phi\Phi}_{+,\mathcal{O}}(u,v) - \sum_{\mathcal{O}} (-1)^{\ell} \lambda^2_{\phi\Phi\mathcal{O}} F^{\Phi\phi;\phi\Phi}_{+,\mathcal{O}}(u,v) = 0$$

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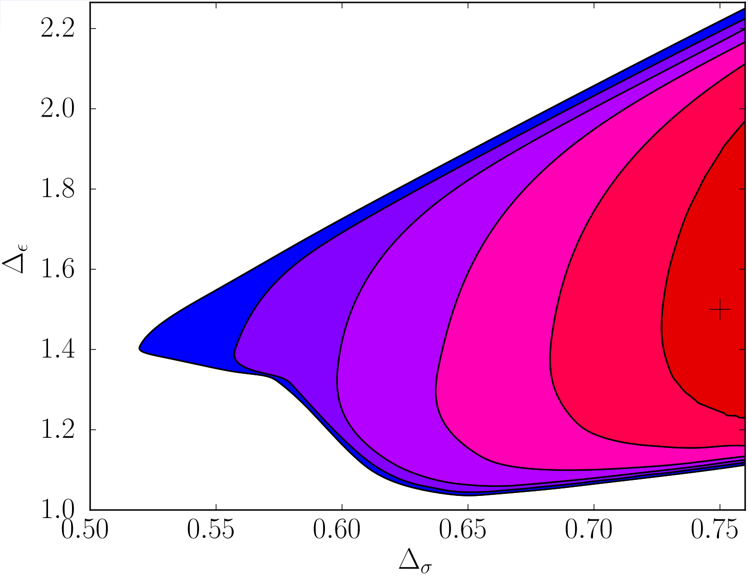
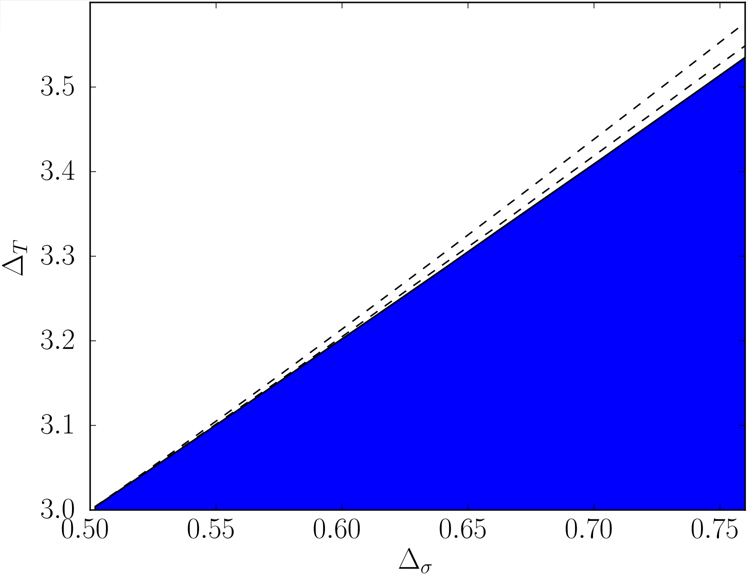
Rule out solutions with SDPB [Simmons-Duffin; 1502.02033] .



$\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle$ with $\Delta_T \geq 3, 3.1, 3.2$ etc.

Red region should move right by $\approx 5\%$.



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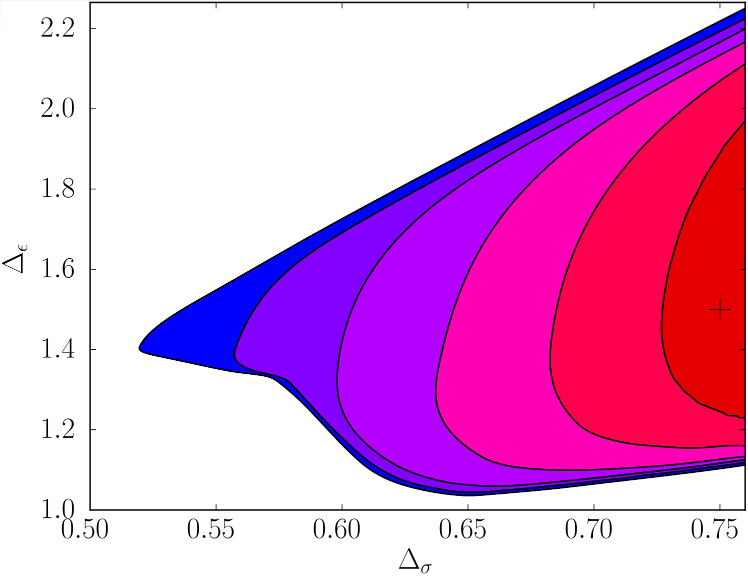

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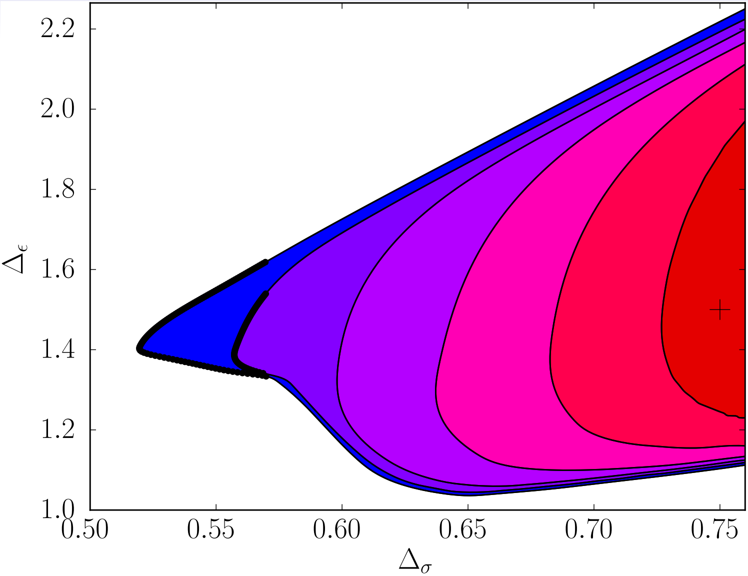

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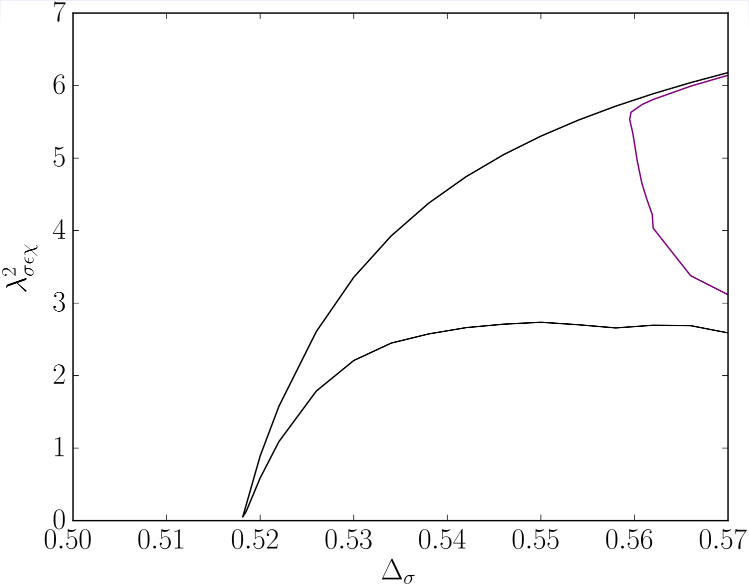
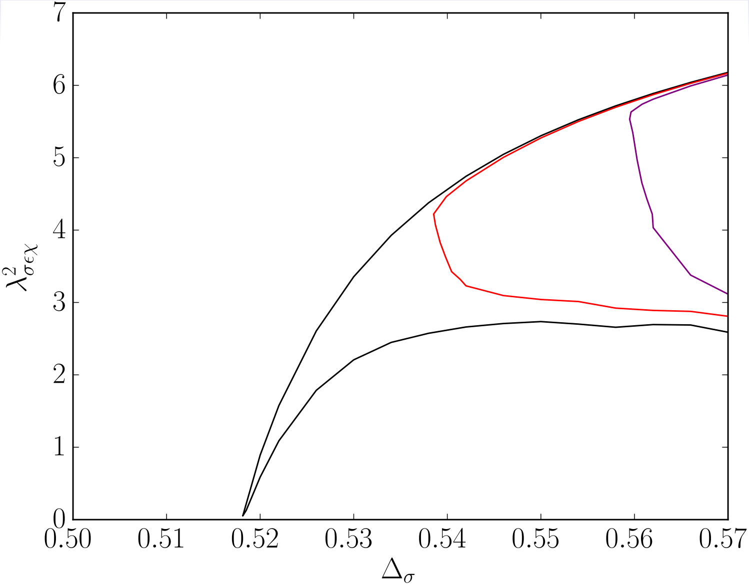


With three correlators, we can see χ decouple at the SRI point.



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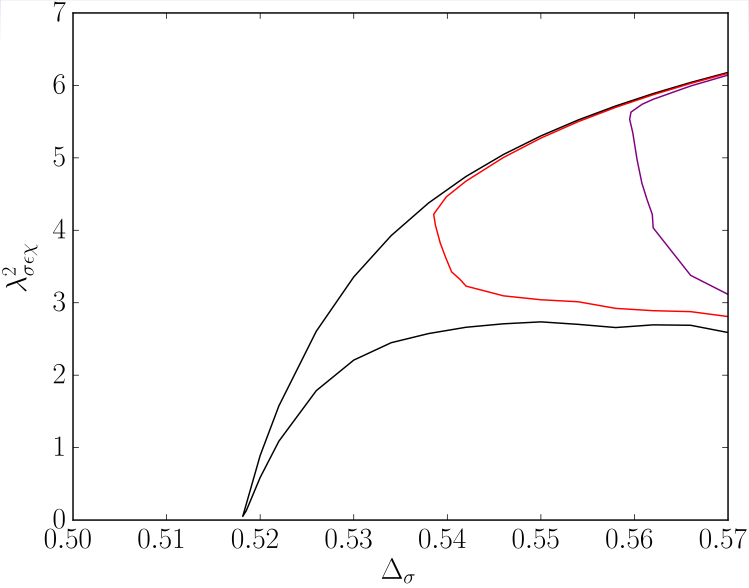

With three correlators, we can see χ decouple at the SRI point.
Should really include $\langle\sigma\sigma\chi\chi\rangle, \langle\epsilon\epsilon\chi\chi\rangle, \langle\chi\chi\chi\chi\rangle$.

The shadow relation

Nonlocal EOM $\chi = g\partial^s\sigma$ expands to

$$n_\chi(s)\chi(x) = \int \frac{n_\sigma(s)\sigma(y)}{|x-y|^{d+s}} dy$$

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Relate 3pt functions with Symanzik: [Paulos, Rychkov, van Rees, Zan; 1509.00008]

$$\frac{\lambda_{12\sigma}}{|x_{10}|^{\Delta_\sigma+\Delta_{12}}|x_{20}|^{\Delta_\sigma-\Delta_{12}}|x_{12}|^{\Delta_1+\Delta_2-\Delta_\sigma}} \quad \mapsto \quad \frac{\lambda_{12\chi}}{|x_{10}|^{\Delta_\chi+\Delta_{12}}|x_{20}|^{\Delta_\chi-\Delta_{12}}|x_{12}|^{\Delta_1+\Delta_2-\Delta_\chi}}$$

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Do this twice to cancel the norms

$$\frac{\lambda_{12\chi}}{\lambda_{12\sigma}} = \frac{n_\sigma(s)}{n_\chi(s)}R_{12} \quad \frac{\lambda_{34\chi}}{\lambda_{34\sigma}} = \frac{n_\sigma(s)}{n_\chi(s)}R_{34}$$

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Result for $\langle\sigma(x_0)\phi_1(x_1)\phi_2(x_2)\rangle$:

$$R_{12} = \pi^{\frac{d}{2}} \frac{\Gamma\left(\Delta_\sigma - \frac{d}{2}\right)\Gamma\left(\frac{\Delta_\chi+\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta_\chi-\Delta_{12}}{2}\right)}{\Gamma\left(\Delta_\chi\right)\Gamma\left(\frac{\Delta_\sigma+\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta_\sigma-\Delta_{12}}{2}\right)}$$

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Embedding space treatment of conformal integrals [Simmons-Duffin; 1204.3894] .

The shadow relation

Interesting choice is $1 = \sigma$, $2 = \epsilon$, $3 = \chi$ and $4 = \epsilon$.

$$\lambda^2_{\sigma\chi\epsilon} = \frac{R_{\sigma\epsilon}}{R_{\chi\epsilon}}\lambda_{\sigma\sigma\epsilon}\lambda_{\chi\chi\epsilon}$$

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For even spin, work with superblocks instead of individual $\lambda_{\sigma\sigma\mathcal{O}} \lambda_{\chi\chi\mathcal{O}} G^{0,0}_{\mathcal{O}}(u, v)$ and $\lambda^2_{\sigma\chi\mathcal{O}} G^{\Delta_{\chi\sigma}, \Delta_{\sigma\chi}}_{\mathcal{O}}(u, v)$.

$$\mathcal{G}_{\mathcal{O}}(u, v) = G^{0,0}_{\mathcal{O}}(u, v) + v^{\Delta_\sigma - \frac{d}{2}} \frac{R_{\sigma\mathcal{O}}}{R_{\chi\mathcal{O}}} G^{\Delta_{\chi\sigma}, \Delta_{\sigma\chi}}_{\mathcal{O}}(u, v)$$

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$$\frac{R_{\sigma\mathcal{O}}}{R_{\chi\mathcal{O}}} = \frac{\Gamma\left(\frac{d-\Delta+\ell}{2}\right)^2 \Gamma\left(\frac{d-2\Delta_\sigma+\Delta+\ell}{2}\right)\Gamma\left(\frac{2\Delta_\sigma-d+\Delta+\ell}{2}\right)}{\Gamma\left(\frac{\Delta+\ell}{2}\right)^2 \Gamma\left(\frac{2\Delta_\sigma-\Delta+\ell}{2}\right)\Gamma\left(\frac{2d-2\Delta_\sigma-\Delta+\ell}{2}\right)}$$

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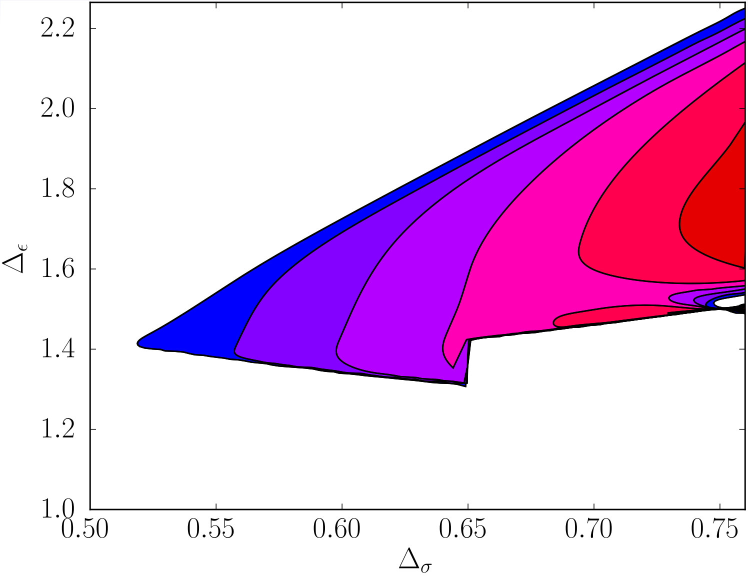
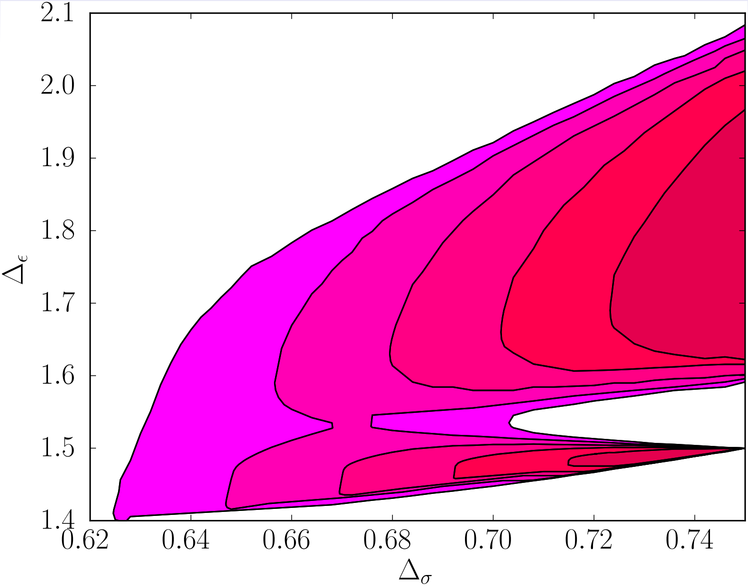
If the dimension of $[\sigma\chi]_{n,\ell}$ ever moves away from the pole, OPE coefficients must jump to zero.

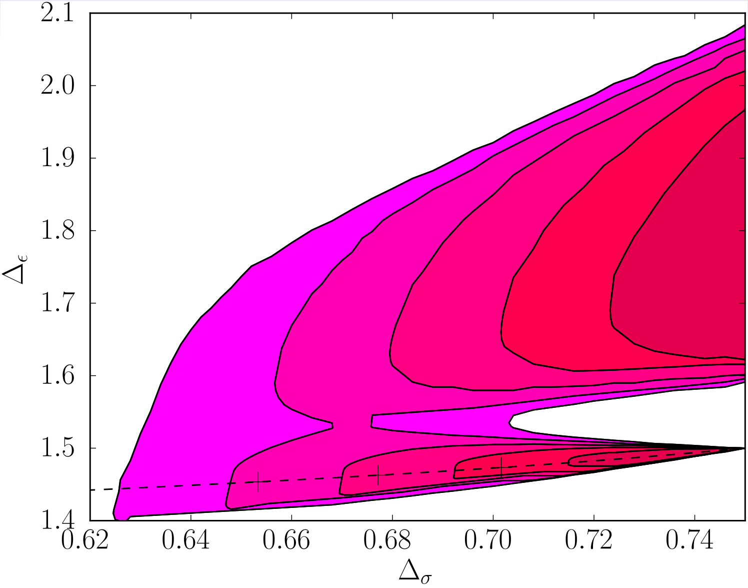


Interesting structure near MFT.



Interesting structure near MFT.

Region further splits into lobes after a higher precision scan.



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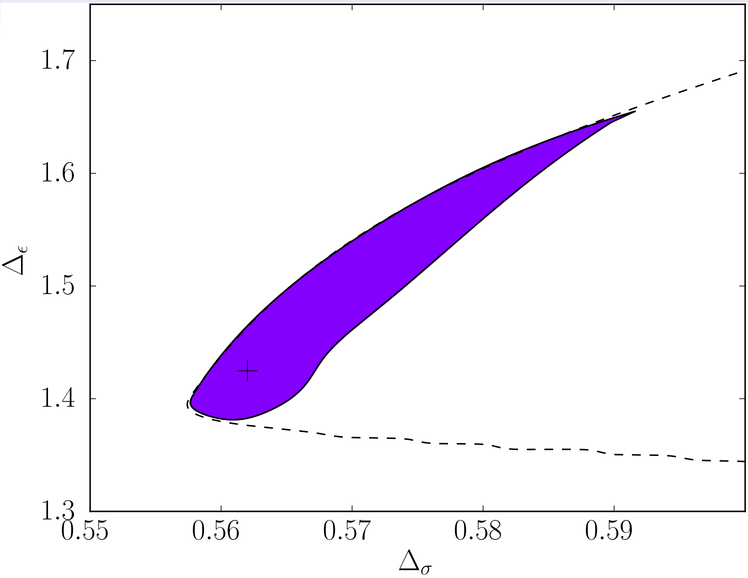

To improve this region, use superblocks and impose another gap: $\Delta_T \in \{3.1\} \cup (4.5, \infty)$ instead of $\Delta_T \in (3.1, \infty)$.

Future improvements

SDPB requires “positive-times-polynomial” form for each block:

$$\partial_z^m \partial_{\bar{z}}^n G_{\Delta,\ell}\left(\frac{1}{2},\frac{1}{2}\right) = \chi_\ell(\Delta) P_\ell^{(m,n)}(\Delta)$$

$$\chi_\ell(\Delta) = \frac{e^{c\Delta}}{\prod_{k=1}^{N}(\Delta - \Delta_k)}$$

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Fine for superblocks: all single poles below unitarity bound ✓.

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Impossible for superblocks: $\Delta = d + \ell + 2n \geq \Delta_{min}$ singularity X.

Future improvements

$$\lim_{\ell \to 2k+1} \frac{\lambda_{\sigma\sigma\mathcal{O}}\lambda_{\chi\chi\mathcal{O}}}{(\Delta - d - \ell - 2n)^2} = ?$$

- We only know dimensions of protected operators so far.
- $\lambda^2_{\sigma\chi\mathcal{O}}$ would be useful too [Beem, Rastelli, van Rees; 1304.1803] .
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